Algorithms Week 1

Ljubomir Perković, DePaul University

Course overview

We will cover techniques for designing computer algorithms, including:

- divide-and-conquer
- backtracking
- dynamic programming
- greedy

We will later use graph algorithms as a case study.

Throughout, we will also make use of techniques for analyzing computer algorithms and problems including, towards the end of the course, the theory of NP-completeness.

Course mechanics

See the course web site https://reed.cs.depaul.edu/lperkovic/courses/csc321

- Find there the course syllabus
- My weekly lecture slides and video recordings will be posted there on Mondays

I will hold optional, recorded Zoom discussion sessions to discuss the week's topics on Wednesdays

- The discussion session recording will be posted on the course web site
- The weekly homework assignment will be posted on Wednesday as well
- The homework assignment will be due the following Wednesday

A final exam will be given at the end of the course.

Contact

Discussion sessions: Wed, 3:00pm-4:30pm CT, Zoom meeting link in D2L

Office hours: Mon, 3:00pm-4:30pm and 7:30pm-9:00pm, Zoom meeting link in D2L

Discussion forum: Discord server link in D2L

E-mail: lperkovic@cs.depaul.edu

Phone: 312-362-8337 (leave a message)

Algorithms

An algorithm is a step-by-step procedure for solving a problem

- Typically developed before doing any programming
- In fact, it is independent of any programming language

Efficient algorithms can have a dramatic effect on our problem-solving capabilities

Algorithm design concerns

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6. (Sometimes) optimality - can we prove that the algorithm does the best of any algorithm?

Polynomial evaluation

We consider the problem of evaluating a polynomial. A precise specification of the problem would be:

Input: A polynomial $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ and a value z. We assume the coefficient are stored in an array $a[0..n]$.

Output: The evaluation of $p(x)$ at z, i.e. $p(z)$.

Example: If the input is $p(x) = x^2 + 2x + 1$ and $z = 3$ then the output should be $p(3) = 16$

Naive solution

We assume that the coefficients $a_0, a_1, a_2, \ldots, a_n$ are stored in an array $a[0..n]$ of size $n + 1$.

```
NaiveEvaluation(a, n, z)
  res ← 0
  for i ← 0 to n
    zpoweri ← 1
    for j ← 1 to i
      zpoweri ← zpoweri * z
    res ← res + a[i] * zpoweri
  return res
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Is it correct?

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Is it correct? Note: To formally prove correctness you need to use mathematical induction

What do we mean by time?

Suppose that we mean the number of lines of pseudocode executed. The question is still imprecise, as the answer will depend on the size of the input.

Let us denote the size of the input by n.

The problem is then to determine the number of lines $T(n)$ executed by our algorithm on a polynomial of degree n.

The number of lines executed is

$$\begin{aligned} 2 + \sum_{i=0}^{n} (3 + 2i) &= 2 \sum_{i=0}^{n} i + 3(n+1) + 2 \\ &= n^2 + 4n + 5 \end{aligned}$$

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OK... So what? What if the number was $3n^2 - 10n + 62$???

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More efficient solution

```
BetterEvaluation(a, n, z)
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The number of lines executed is

$$3 + \sum_{i=0}^{n} 3 = 3n + 6$$

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The number of lines executed is

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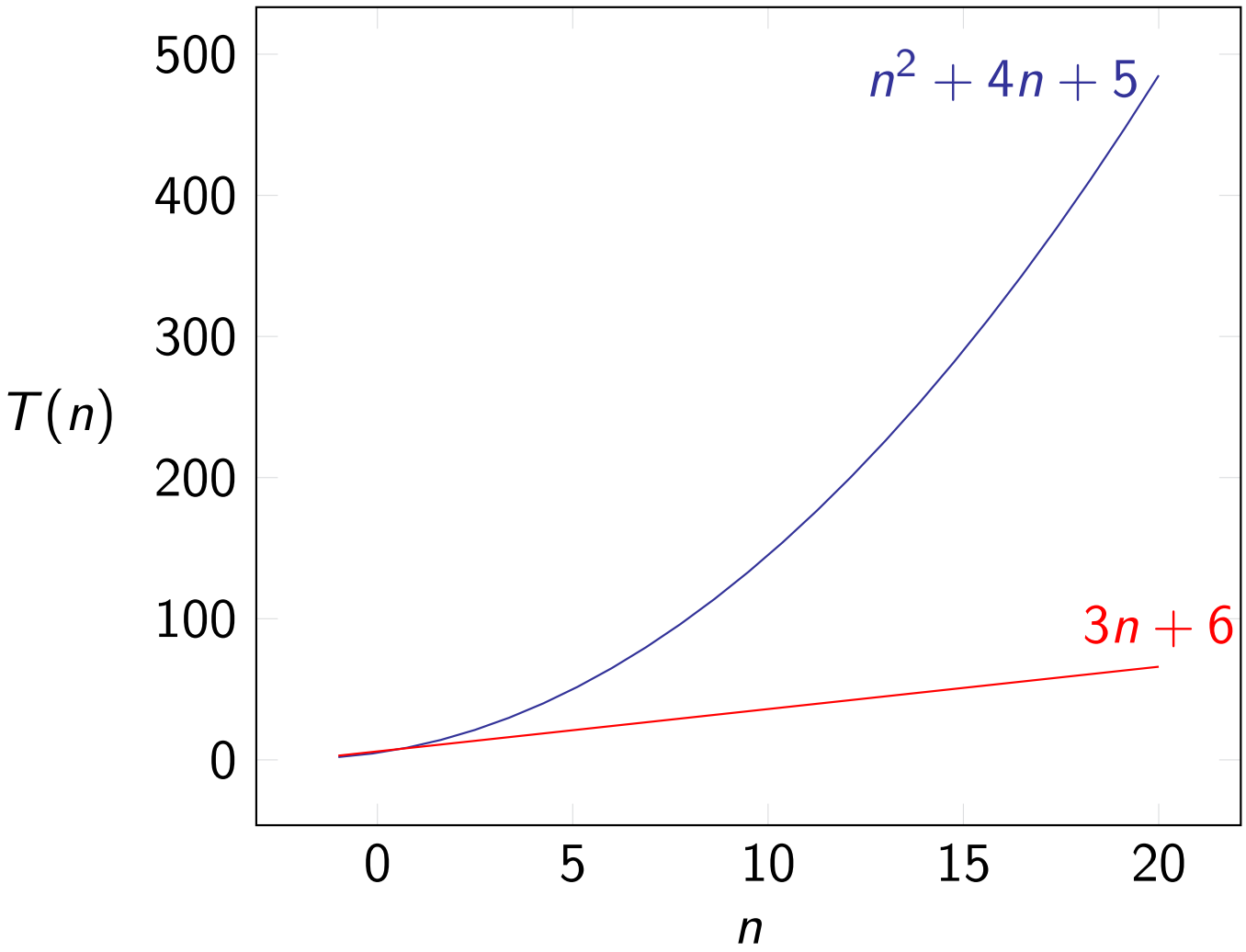
which is much less than $n^2 + 4n + 5$ when n gets large

Running time comparison



Note that the comparison of the two functions boils down to the a comparison of the two functions' growth rates

Consider the function $T(n) = n^2 + 4n + 5$ that we produced as the running time of the naive polynomial evaluation algorithm.

We can approximate the behavior of an algorithm by considering only the highest order term in the function T(n). This is because as the size of the problem gets larger, the fastest growing term represents the corresponding growth of the running time.

In this course we will only be interested in this asymptotic behavior of algorithms. This motivates us to define a notation that will make the analysis of algorithms simpler.

Big-O notation

We will be interested in three types of asymptotic behavior.

O notation is used to bound the asymptotic behavior of a function from above (upper bound).

Definition

Let f and g be (non-negative) functions. If there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$ then we say that $f(n)$ is $O(g(n))$, typically denoted by $f(n) = O(g(n))$.

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- Does this mean that $f(n)$ is always $\leq g(n)$? No!
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- It means that $f(n)$ is eventually $\leq cg(n)$, for large enough values of n.

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- Does it mean that $f(n)$ is always $\leq cg(n)$? No!
- It means that $f(n)$ is eventually $\leq cg(n)$, for large enough values of n.
- Big-O notation is the one we will use 95% of the time

Example

Consider the function $T(n) = n^2 + 4n + 5$ that we produced as the running time of the naive polynomial evaluation algorithm.

Claim

$T(n) = O(n^2)$

Proof.

Choose $c = 10$ and $n_0 = 1$. We verify that for all $n \geq 1$, $n^2 + 4n + 5 \leq 10n^2$:

$$\begin{aligned} n^2 + 4n + 5 &\leq n^2 + 4n^2 + 5n^2 \text{ (since } n \geq 1) \\ &\leq 10n^2 \end{aligned}$$

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Note: We could have chosen $c = 2$ and $n_0 = 5$ but the above choices leads to a simpler argument

Example

Consider the function $T(n) = 3n + 6$ that we produced as the running time of the better polynomial evaluation algorithm.

Claim

$T(n) = O(n)$

Proof.

Choose $c = 9$ and $n_0 = 1$. We verify that for all $n \geq 1$, $3n + 6 \leq 9n$:

$$\begin{aligned} 3n + 6 &\leq 3n + 6n \text{ (since } n \geq 1) \\ &\leq 9n \end{aligned}$$

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$$\begin{aligned} 3n + 6 &\leq 3n + 6n \text{ (since } n \geq 1) \\ &\leq 9n \end{aligned}$$

□

Note: Again, we could have chosen $c = 4$ and $n_0 = 6$ but the above choices leads to a simpler argument

Example

Consider the function $T(n) = 3n + 6$ that we produced as the running time of the better polynomial evaluation algorithm.

Claim

$T(n) = O(n^2)$

Proof.

Left as an exercise $\square$

Example

Consider again the function $T(n) = n^2 + 4n + 5$ that we produced as the running time of the naive polynomial evaluation algorithm.

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$T(n) = O(n)$

No, that is not true!

If it were true, it would mean that there exist positive constants c and n_0 such that $n^2 + 4n + 5 \leq cn$ for all $n \geq n_0$.

However, when $n \geq c$ then $n^2 + 4n + 5 \geq cn + 4n + 5 > cn$

So, for $n \geq \max\{n_0, c\}$ we get that $n^2 + 4n + 5 \leq cn$ and that $n^2 + 4n + 5 > cn$ which is a contradiction

Big-Omega

Ω notation is used to bound the asymptotic behavior of a function from below (lower bound).

Definition

Let f and g be (non-negative) functions. If there exist positive constants c and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$ then we say that $f(n)$ is $\Omega(g(n))$, denoted by $f(n) = \Omega(g(n))$.

It means that $f(n)$ is eventually $\geq cg(n)$.

Consider again the function $T(n) = n^2 + 4n + 5$ that we produced as the running time of the naive polynomial evaluation algorithm.

Claim

$T(n) = \Omega(n^2)$.

Proof.

Choose $c = 1$ and $n_0 = 1$. We easily verify that for all $n \geq 1$

$$n^2 + 4n + 5 \geq n^2$$

□

So $T(n) = O(n^2)$ and $T(n) = \Omega(n^2)$

Big-Theta

Θ notation is used to capture exactly the asymptotic behavior of a function

Definition

Let f and g be (non-negative) functions. If there exist positive constants c_1, c_2 and n_0 such that for all $n \geq n_0$, $c_1 g(n) \leq f(n) \leq c_2 g(n)$ then we say that that $f(n)$ is $\Theta(g(n))$, denoted $f(n) = \Theta(g(n))$.

In other words,

$f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n)$ and $f(n) = \Omega(g(n))$,

and $g(n)$ is said to be an asymptotically tight bound on the function f(n)

Claim (Transitivity)

If $f(n) = O(g(n))$ *and* $g(n) = O(h(n))$ *then* $f(n) = O(h(n))$.

Note: This also holds for Ω and Θ.

Proof.

If $f(n) = O(g(n))$ then there exist c_1, n_1 s.t. for all $n \geq n_1$, $f(n) \leq c_1 g(n)$.

If $g(n) = O(h(n))$ then there exist c_2, n_2 s.t. for all $n \geq n_2$, $g(n) \leq c_2 h(n)$.

Let $n3 = max\{n1, n2\}$ and let $c_3 = c1 * c2$. Then for all $n \geq n_3$, $f(n) \leq c_1 g(n) \leq c_1 c_2 h(n) = c3h(n)$. Thus $f(n) = O(h(n))$. □

Claim

If $f_1(n) = O(g_1(n))$, $f_2(n) = O(g_2(n))$, *and* $g_1(n) = O(g_2(n))$ *then* $f_1(n) + f_2(n) = O(g_2(n))$.

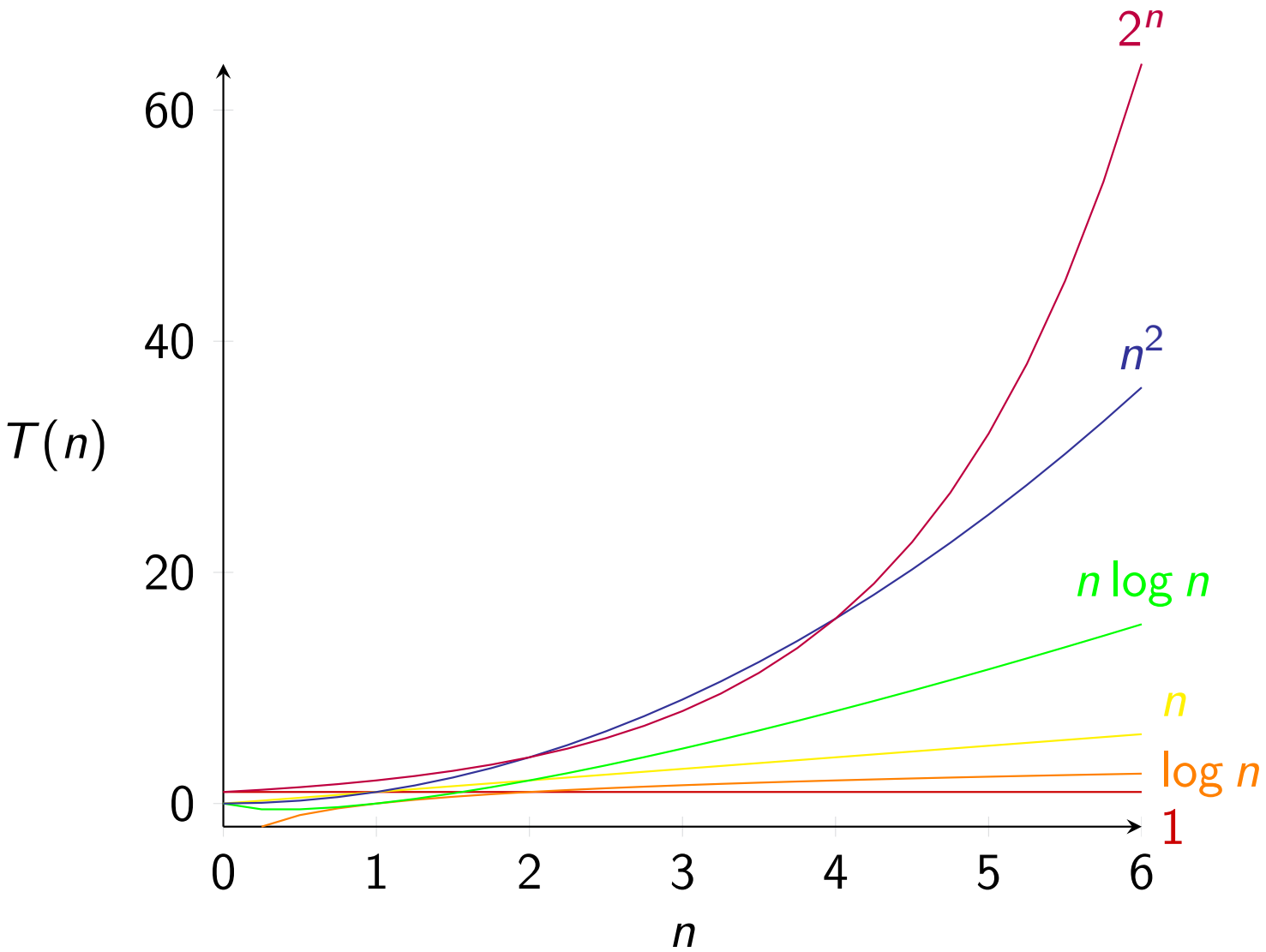
The proof is left as a homework problem.

Exercise

Compare the asymptotic growth rates of the following functions:

$$1, \lg n, n, n \lg n, n^2, 2^n.$$

Exercise (cont.)



Consider again our algorithm for polynomial evaluation:

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Is it optimal?

Horner's method

By rewriting $p(x)$ as

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \\ &= a_0 + x(a_1 + x(a_2 + x(a_3 + x(\ldots(a_{n-1} + x(a_n + 0))\ldots)))) \end{aligned}$$

we develop an alternate algorithm:

```
HornersMethod(a, n, z)
res ← 0
for i ← n downto 0
  res ← res * z + a[i]
output res
```

The number of lines executed by Horner's method is

$$2 + \sum_{i=0}^{n} 2 = 2 + 2(n+1) = 2n + 4$$

which is less than $3n + 6$.

Optimality

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Optimality

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```

Is it optimal?

Horner's algorithm is clearly better but the running time of both algorithms is $O(n)$

Because any polynomial evaluation algorithm must run in time $\Omega(n)$ (why?), Horner's and BetterEvaluation algorithms are both asymptotically optimal

Course overview: Describing algorithms

A complete description of any algorithm has four components:

What: A precise statement of the problem.

How: A precise description of the algorithm.

Why: A proof that the algorithm is correct, i.e. it solves the problem it is supposed to solve.

How fast: An analysis of the running time of the algorithm, also known as *asymptotic analysis*.

Course overview: Describing algorithms

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- **What:** A precise statement of the problem.
- **How:** A precise description of the algorithm.
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- **How fast:** An analysis of the running time of the algorithm, also known as *asymptotic analysis*.

Problem: Input size is not the only parameter that determines running time

A fundamental problem is sorting an array of numbers.

Input: A array $a[0..n-1]$ of n numbers

Output: A re-ordering of the numbers in the array such that $a[0] \leq a[1] \leq \cdots \leq a[n-1]$

There are many different ways of approaching the problem

InsertionSort works by repeatedly moving elements into their proper relative positions

The steps of the algorithm are:

1. Move $a[1]$ to the left until all elements to its left are no greater than it
2. Move $a[2]$ to the left until all elements to its left are no greater than it
3. Repeat up until element $a[n-1]$

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Note: After i iterations, the first $i+1$ elements of the list will be in sorted order

InsertionSort example

For example, let the input list of numbers be 5, 2, 4, 6, 1, 3. Then the iterations of the algorithm will be the following:

Initial order:	5	2	4	6	1	3
After 1st iteration:	2	5	4	6	1	3
After 2nd iteration:	2	4	5	6	1	3
After 3rd iteration:	2	4	5	6	1	3
After 4th iteration:	1	2	4	5	6	3
After 5th iteration:	1	2	3	4	5	6

If a[0..n-1] is an array of numbers to be sorted, the pseudocode for insertion sort is:

```
InsertionSort(a, n)
for j ← 1 to n-1
  key ← a[j]
  i ← j-1
  while i ≥ 0 and a[i] > key
    a[i+1] ← a[i]
    i ← i-1
  a[i+1] ← key
```

InsertionSort running time analysis

Because we are only interested in the asymptotic behavior of the algorithm, we need only focus on an operation in the innermost loop, e.g. the while loop condition:

$$T(n) = \sum_{j=1}^{n-1} \sum_{?} 1.$$

The first summation counts the number of iterations of the for-loop, and the second summation counts the number of times the while-loop condition is evaluated within a single iteration of the outer for-loop

We know that the for-loop will be executed $n - 1$ times.

What about the number of times the while-loop condition will be evaluated?

InsertionSort running time analysis

Suppose that the list contains 1 2 3 4 5 6. Then the execution will look like the following:

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InsertionSort running time analysis

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Initial order:	1	2	3	4	5	6
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Suppose that the list contains 1 2 3 4 5 6. Then the execution will look like the following:

Initial order:	1	2	3	4	5	6
After 1st iteration:	1	2	3	4	5	6
After 2nd iteration:	1	2	3	4	5	6
After 3rd iteration:	1	2	3	4	5	6
After 4th iteration:	1	2	3	4	5	6
After 5th iteration:	1	2	3	4	5	6

Note that the while-loop condition is evaluated only once in every iteration of the outer for-loop and so

$$T(n) = \sum_{j=1}^{n-1} 1 = O(n)$$

InsertionSort running time analysis

Suppose that the list contains 6 5 4 3 2 1. Then the execution will look like the following:

Initial order: 6 5 4 3 2 1

InsertionSort running time analysis

Suppose that the list contains 6 5 4 3 2 1. Then the execution will look like the following:

Initial order:	6	5	4	3	2	1
After 1st iteration:	5	6	4	3	2	1

InsertionSort running time analysis

Suppose that the list contains 6 5 4 3 2 1. Then the execution will look like the following:

Initial order:	6	5	4	3	2	1
After 1st iteration:	5	6	4	3	2	1
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After 5th iteration:	1	2	3	4	5	6

Within iteration j of the outer for-loop, the while-loop condition is evaluated for $i = j - 1, j - 2, \ldots, 0$, i.e. a total of j times. Thus

$$T(n) = \sum_{j=1}^{n-1} j = \frac{n(n-1)}{2} = O(n^2)$$

InsertionSort running time analysis

The running time of an algorithm can depend on the type of input it is given. Some inputs are "harder" than others for an algorithm. There are three different approaches to analyzing running times:

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Average-case This gives the average (or expected) number of steps over all possible inputs to the algorithm. In order to be computed, a probability distribution on the inputs must be assumed.

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InsertionSort average-case analysis

If we assume that any ordering is equally likely, then we can argue that we expect that each `a[j]` moves $\frac{1}{2}(j-1)$ positions to the left. In other words, the number of times the while-loop condition is evaluated in each iteration of the outer for-loop is $\frac{1}{2}j$. The expected number of steps is then

$$T(n) = \sum_{j=1}^{n-1} \frac{1}{2} j = \frac{n(n-1)}{4} = O(n^2)$$

Course overview: core techniques

Algorithm design techniques that we will use in this class include:

1. divide-and-conquer
2. backtracking
3. dynamic programming
4. greedy

All of the above techniques rely on reductions.

Reductions

As your textbook says:
"Reductions are the single most common technique used in designing algorithms.

Reducing one problem X to another problem Y means to write an algorithm for X that uses an algorithm for Y as a black box or subroutine.

Crucially, the correctness of the resulting algorithm for X cannot depend in any way on how the algorithm for Y works. The only thing we can assume is that the black box solves Y correctly. The inner workings of the black box are simply none of our business; they're somebody else's problem. It's often best to literally think of the black box as functioning purely by magic."

Reduction examples

1. The problem of evaluating a polynomial is reduced to the basic operations of addition and subtraction
2. In InsertionSort, the problem of sorting is reduced to the (easier) problem of inserting a number into an already sorted list
3. In your textbook, the Huntington-Hill algorithm reduces the problem of apportioning Congress to the problem of maintaining a priority queue that supports the operations Insert and ExtractMax.

Recursion

Recursion is a special type of reduction that reduces a problem instance to one or more simpler instances of the same problem.

All four algorithm design techniques

1. divide-and-conquer
2. backtracking
3. dynamic programming
4. greedy

that we will use in this class can be described using recursion.

Divide-and-conquer approach

A divide-and-conquer algorithm works as follows:

1. If the problem is small enough, solve it directly (and quickly).
2. Otherwise, divide it into subproblems **(Divide)** that your solve recursively ("as a black box functioning purely by magic"!)
3. Combine the solutions to the subproblems into a solution of the original problem **(Conquer)**

Exponentiation

Consider the problem of computing a^n:

Input: number a and positive integer n

Output: a^n

Example: given $a = 1.5$ and $n = 2$, $a^n = 1.5^2 = 2.25$

The obvious algorithm:

```
SlowPower(a, n)
  res ← a
  for i ← 2 to n
    res ← res * a
  return res
```

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How could divide-and-conquer possibly help?

Fast exponentiation

Indian scholar Pingala proposed the following recursive formula in 2nd century BCE!

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ (a^{n/2})^2 & \text{if } n > 0 \text{ and n is even} \\ (a^{\lfloor n/2 \rfloor})^2 a & \text{otherwise} \end{cases}$$

which leads to the following exponentiation algorithm:

```
PingalaPower(a, n)
  if n ← 1
    return a
  tmp ← PingalaPower(a, ⌊n/2⌋)
  if n is even
    return tmp*tmp
  else
    return tmp*tmp*a
```

Fast exponentiation

Why is it said to be fast?

```
PingalaPower(a, n)
  if n ← 1
    return a
  tmp ← PingalaPower(a, ⌊n/2⌋)
  if n is even
    return tmp*tmp
  else
    return tmp*tmp*a
```

Fast exponentiation

Why is it said to be fast?

How many multiplications are performed by this algorithm?

```
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  if n is even
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