Algorithms Week 3

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One way to solve a maze is to try every potential solution in the search space, i.e. every possible sequence of steps from the maze entrance until the maze exit is found.

Backtracking systematizes the search using the following rule: when we hit a dead end, we back up until we find a place where a choice we have not tried can be made and we make that choice.

Backtracking, more formally

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- trying another extension if the extension fails,
- backing up to a smaller partial solution when the options for extending a partial solution are exhausted.

n-Queens problem

Classic problem to illustrate backtracking.

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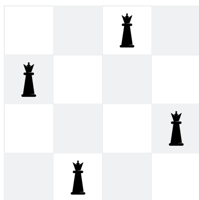
A solution for$n = 4$:



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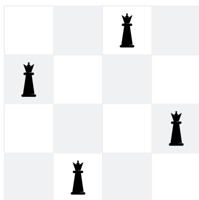

This solution can be represented as $[(1,3),(2,1),(3,4),(4,2)]$.

In order to use backtracking for this problem, we need to describe what a potential solution looks like.

As seen in the previous example, a potential solution is n points in an $n \times n$ array:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

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Note that an exhaustive search will check $(n^2)^n$ possible solutions, one for each set of n coordinates.

Let us use backtracking to narrow down the search space. To do this, we first need to design a more efficient representation of potential solutions and to develop tests that check whether a partial solution can be extended to a complete solution.

We can make several observations that eliminate some of the possible partial solutions:

1. There must be exactly one queen per row; this means that the solutions must have the form:

$$(1, y_1), (2, y_2), \ldots, (n, y_n).$$

Analyzing the partial solutions and pruning

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2. There must be exactly one queen per column; this means that $(y_1, y_2, \ldots, y_n)$ must be a permutation of $(1, 2, \ldots, n)$. Note that this decreases the search space to "only" $n!$ permutations.

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3. There is at most one queen on each diagonal; this means that if two queens are at positions (a, b) and (c, d) then $|a - c| \neq |b - d|$.

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3. There is at most one queen on each diagonal; this means that if two queens are at positions (a, b) and (c, d) then $|a - c| \neq |b - d|$.

We say that we prune a partial solution if we do not extend it any further because it cannot be extended to a complete solution.

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We will generate all $n!$ permutation until we find a solution. We do this by building up the solutions one position at a time.

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Step i: Given the positions of the first $i-1$ queens $(1, y_1), (2, y_2), \ldots, (i-1, y_{i-1})$, choose the position for (i, y_i).

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If you cannot complete Step i for some choice of previous positions $(1, y_1), (2, y_2), \ldots, (i-1, y_{i-1})$, then backtrack and re-choose the position for $(i-1, y_{i-1})$.

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If you cannot complete Step i for some choice of previous positions $(1, y_1), (2, y_2), \ldots, (i-1, y_{i-1})$, then backtrack and re-choose the position for $(i-1, y_{i-1})$.

We will prune a partial solution if it forces the placement of two queens on the same column or diagonal.

n-Queens backtracking algorithm

Let $Q[1 \dots n]$ store the positions of the n queens, i.e. $Q[i]$ will be the column of the queen in row i.

```
PlaceQueens(Q[1 .. n], r):
  // extends a partial solution of size r-1
  // to one of size r,
  // unless the partial solution
  // is a complete solution, in
  // which case it is output;
  // the initial call is PlaceQueens(Q, 1)
  // with Q storing a partial solution of size 0







.
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PlaceQueens(Q[1 .. n], r):
  if r = n + 1
    print Q[1 .. n]
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PlaceQueens(Q[1 .. n], r):
  if r = n + 1
    print Q[1 .. n]
  else
    for j ← 1 to n
      Q[r] ← j
      if Feasible(Q, r)
        PlaceQueens(Q, r + 1)

Feasible(Q[1 .. n], r)
  // Return true if
  // the placement of the queen
  // in row r is feasible,
  // false otherwise
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Feasible(Q[1 .. n], r)
  for i ← 1 to r-1
    if Q[i] = Q[r] or |Q[i] - Q[r]| = |i-r| then
      return false
  return true
```

Backtracking is a very general-purpose algorithm design technique.

Most game-playing programs use a version of bactracking to decide on the next move

For example, one can develop a relatively simple backtracking algorithm that can play a two-player game perfectly (assuming a game that has no randomness or hidden information and that ends after a finite number of moves).

Given the current state of the game, this backtracking algorithm can tell you whether it is possible to win against another perfect player and also tell you how to win.

This algorithm takes as input the state of the game X and one of the two players and returns Good or Bad depending on whether player can win or not.

```
PlayAnyGame(X, player):
  if player has already won in state X
    return Good
  if player has already lost in state X
    return Bad
  for all legal moves X ⇝ Y
    if PlayAnyGame(Y, ¬ player) = Bad
      return Good
  return Bad
```

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Input: A set S of positive integers and an integer T.

Output: A subset X of S such that $\sum_{x \in X} x = T$.

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- $6 + 3 + 1 = 10$, so $X = \{6, 3, 1\}$ is a valid solution.

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- $6 + 3 + 1 = 10$, so $X = \{6, 3, 1\}$ is a valid solution.
- $4 + 3 + 1 = 8$, so $X = \{4, 3, 1\}$ is **not** a valid solution.

Recursive solution

Suppose that X is a solution. In other words, $\sum_{x \in X} x = T$.

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In both cases we reduce the problem with inputs S and T to two smaller problem instances: one with inputs $S \setminus \{e\}$ and $T - e$ and one with inputs $S \setminus \{e\}$ and T.

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In other words the solution for inputs S and T can be constructed from the solutions to the problem with inputs $S \setminus \{e\}$ and $T - e$ and to the problem with inputs $S \setminus \{e\}$ and T.

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So, recursion can be used to implement an algorithm.

Backtracking makes use of the recursion to more systematically search the space of potential solutions and prune the search when we find partial solutions that cannot be extended to a complete valid solution.

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Backtracking approach

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We then use array $X[1..r]$ containing zeros and ones to represent a partial potential solution that incorporates our decisions whether to include numbers $S[1], S[2], \ldots, S[r]$, where

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When $r = n$ then the partial potential solution is a complete potential solution.

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- $X = [1, 0, 1, 0, 0]$ represents a partial solution that incorporates our decisions on all numbers in $S[1..5]$; it happens to be a complete and valid solution.
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4. Backtrack if both choices fail to lead to a feasible solution.

Let us implement this strategy on input $S = [4, 7, 6, 3, 1]$ and $T = 10$.

```
SubsetSum(S[1..n], T, X[1..n], r)
  // extends a partial solution X[1..r-1]
  // of size r-1 to one of size r,
  // unless the partial solution
  // is a complete solution, in
  // which case it is output;
  // the initial call is SubsetSum(S, T, X, 1)
  // with X storing a partial solution of size 0

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```

```
SubsetSum(S[1..n], T, X[1..n], r)
  if r = n+1 then
    output X
    return
  X[r] ← 0
  if Feasible(S, T, X, r) then
    SubsetSum(S, T, X, r+1)
  X[r] ← 1
  if Feasible(S, T, X, r) then
    SubsetSum(S, T, X, r+1)
```

Subset Sum algorithm

When is partial solution $X[1..r]$ infeasible?

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```
Feasible(S[1..n], T, X[1..n], r)
  S1 ← ∑_{i=1}^{r} X[i]S[i]
  if S1 > T then
    return false
  return true
```

Furthermore, X can also be pruned if

$$\sum_{i=1}^{k} X[i]S[i] + \sum_{i=r+1}^{n} S[i] < T.$$

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Feasible(S[1..n], T, X[1..n], r)
  S1 ← ∑_{i=1}^{r} X[i]S[i]
  if S1 > T then
    return false
  S2 ← ∑_{i=r+1}^{n} S[i]
  if S1 + S2 < T then
    return false
  return true
```

General recursive backtracking

```
Solve(partial solution)
  if partial solution is a solution then
    // if goal is to find a solution
    output solution and quit
  for all possible extensions of partial solution
    if extension is feasible then
      Solve(extension)
```

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  if partial solution is a solution then
    // if goal is to find all solutions
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  for all possible extensions of partial solution
    if extension is feasible then
      Solve(extension)
```

```
Solve(partial solution)
  if partial solution is a solution then
    // if goal is to find optimal solution
    keep solution if best so far and return
  for all possible extensions of partial solution
    if extension is feasible then
      Solve(extension)
```

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Algorithm engineering is an area of computer science that goes beyond the design and analysis of algorithms and includes their *implementation, optimization, profiling and experimental evaluation*.

To illustrate this, I develop next a backtracking solution for the Kattis problem Class Picture.