Algorithms Week 7

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For example: consider a file that contains 100,000 characters taken from the set $\{a, b, c, d, e, f\}$.

One approach is to store the file using the ASCII encoding. That requires 8 bits per character. Thus the entire file would require 800,000 bits.

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c	010
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The file would then require a total of 300,000 bits and a compression ration of 63% is achieved.

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... because using only two bits gives us four possible codes, but the file contains six different symbols to encode.

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If we represent the more frequent symbols using fewer bits, then the number of bits needed to encode the entire file might decrease.

[1] in thousands

Suppose we use the following encoding:

Symbol	Code	Frequency[2] x Cost
a	0	45 x 1
b	101	13 x 3
c	100	12 x 3
d	111	16 x 3
e	1101	9 x 4
f	1100	5 x 4

[2] in thousands

Suppose we use the following encoding:

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Then encoding the file would require 224,000 bits.

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Then encoding the file would require 224,000 bits. This is 72% fewer bits than the ASCII encoding and 25% fewer bits than the 3-bit fixed length code.

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There is a problem! Decoding "01" is ambiguous because the code for the symbol "a" is a prefix of the code for the symbol "c".

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Note that 0 is not a prefix of any other code, 100 is not a prefix of any other code etc.

Decoding revisited

Symbol	Code
a	0
b	101
c	100
d	111
e	1101
f	1100

Prefix-free codes allow us to easily decode. For example, decoding "010110001100..." gives:

$$\begin{array}{ccccccccccc} 0 & | & 101 & | & 100 & | & 0 & | & 1100 & | \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

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With a prefix-free code, as soon as we see a set of symbols that corresponds to a code, we can decode the code.

Representing a prefix-free code

A prefix-free code can be represented using a full binary tree (a binary tree in which every node is either a leaf or has two children):

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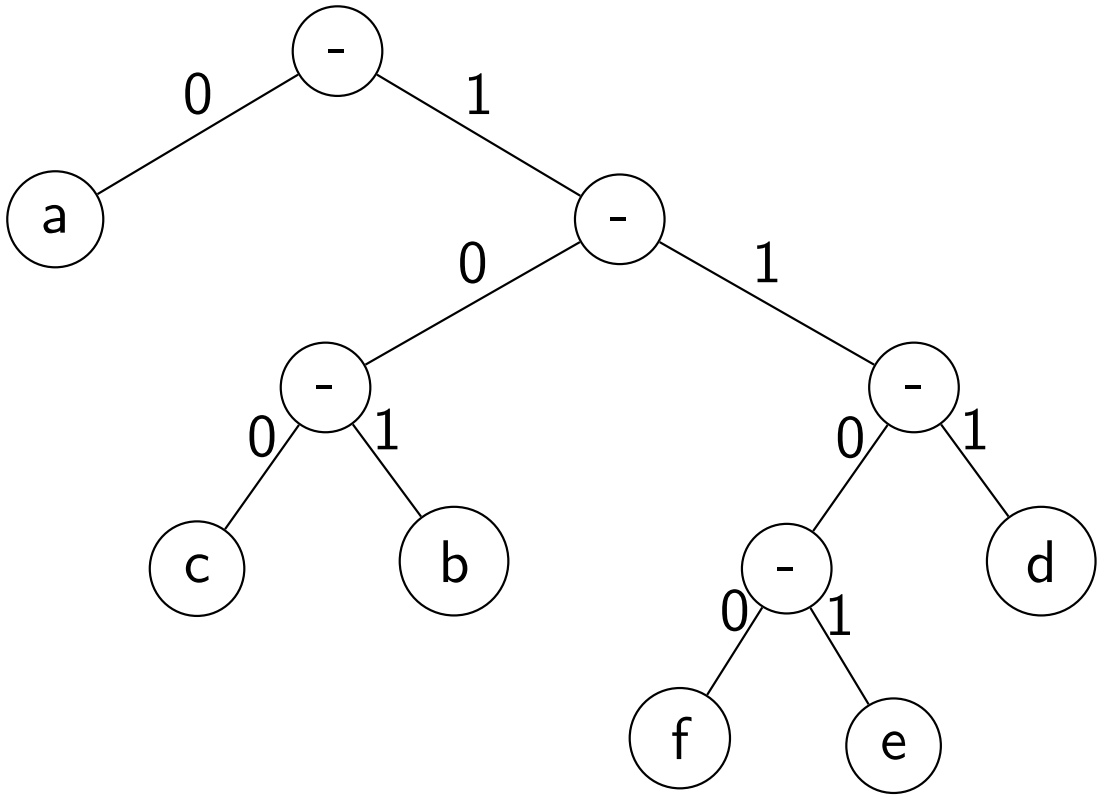
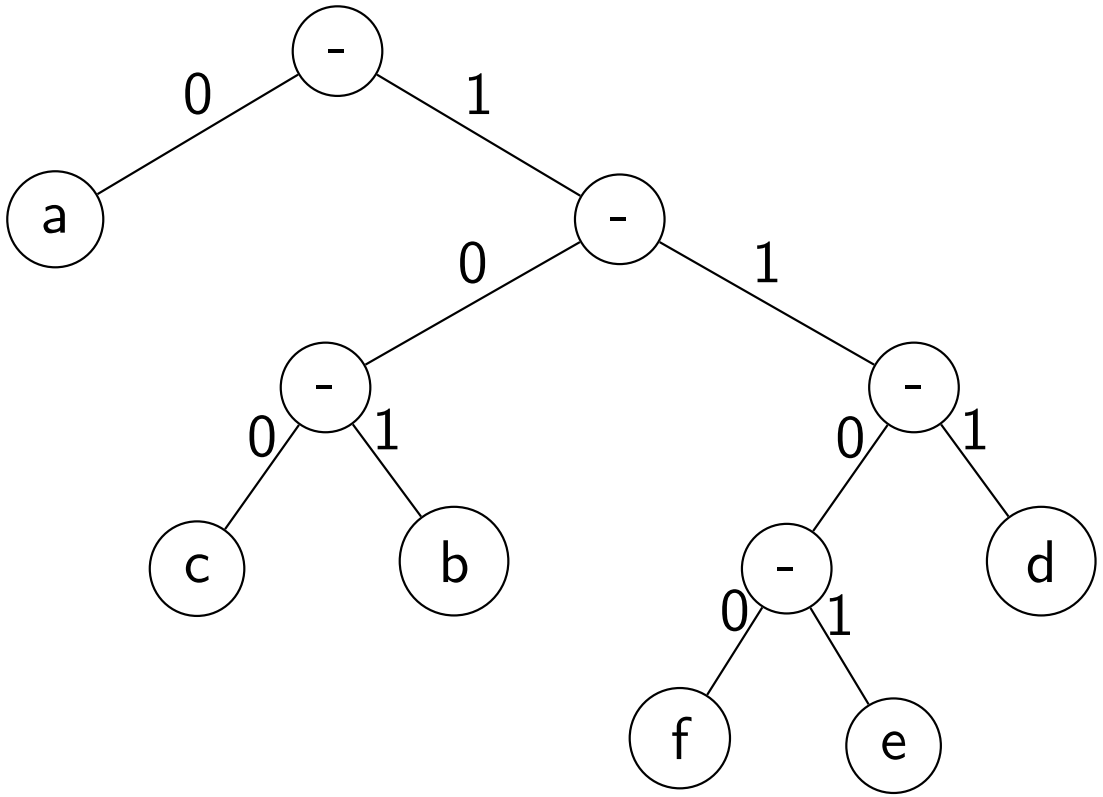


Representing a prefix-free code

A prefix-free code can be represented using a full binary tree (a binary tree in which every node is either a leaf or has two children):

- Each leaf of the tree corresponds to a symbol to be encoded,
- The left and right links from each nonleaf node correspond to 0 and 1, respectively.



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Input: Characters $c[1], c[2], ..., c[n]$ and corresponding frequencies $f[1], f[2], ..., f[n]$.

Output: A tree representing the optimal prefix-free code.

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Output: A tree representing the optimal prefix-free code.

If $d_T(i)$ is the depth of the node corresponding to character $c[i]$ in the tree T, our goal is to find a prefix-free code tree that optimizes the function:

$$B(T) = \sum_{i=1}^{n} f[i] \cdot d_T(i)$$

Huffman's greedy algorithm

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1. Create a tree node for each character and put them in a container.
2. Repeat until there is only one tree in the container:
   a. Greedily merge the two least-frequent trees into a sub-tree whose frequency is equal to the sum of the two tree frequencies.
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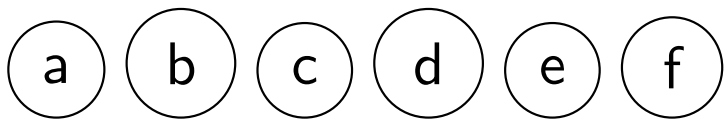

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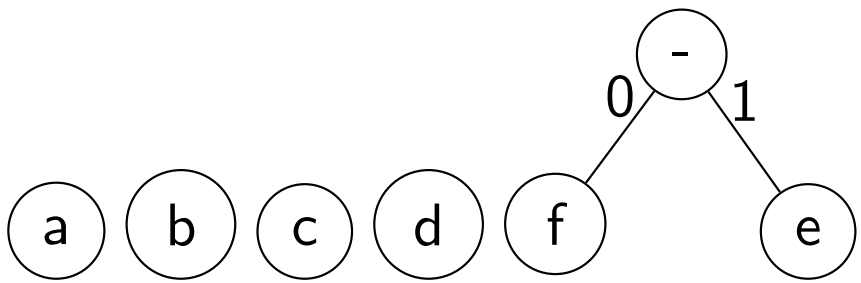

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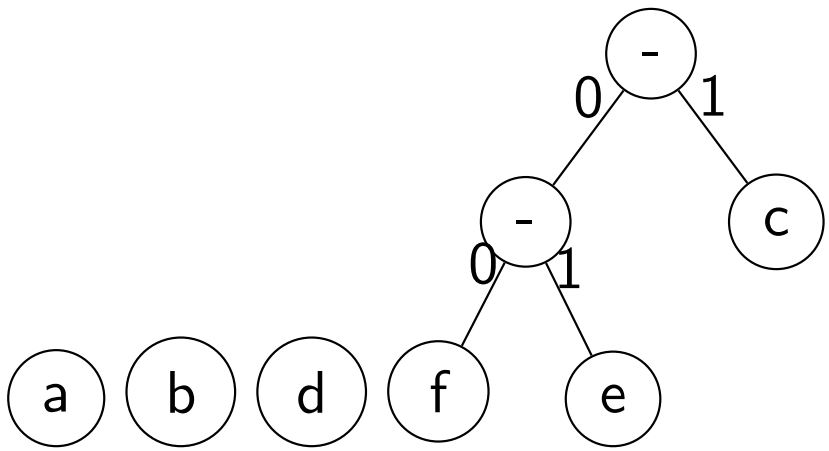

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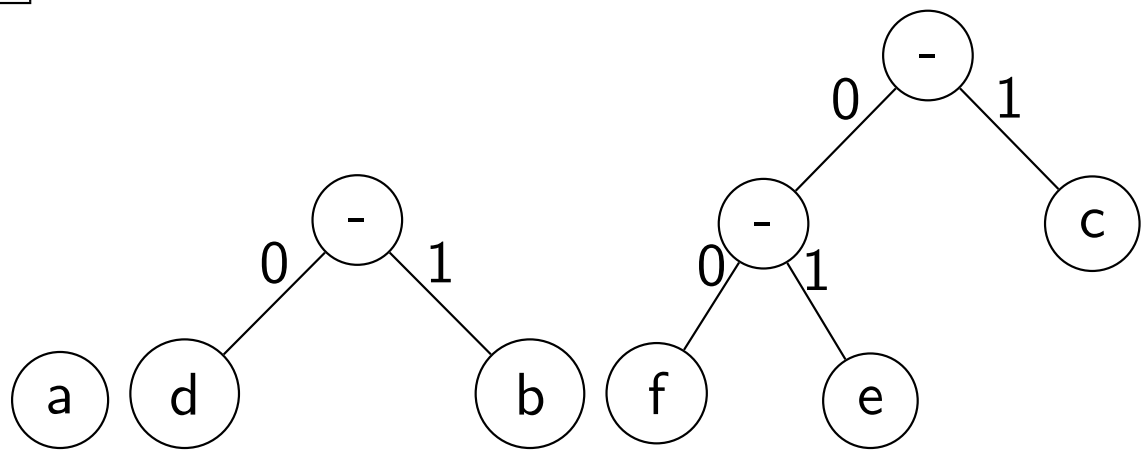
c,e,f	26
b,d	29



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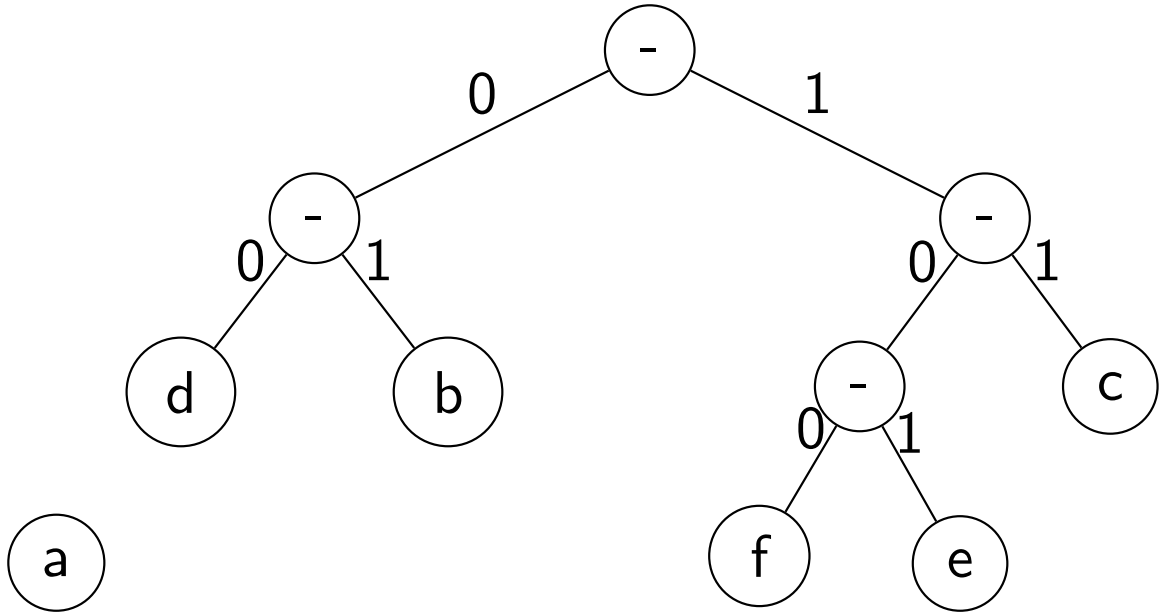
Symbol	Frequency
a	45
b,c,d,e,f	55



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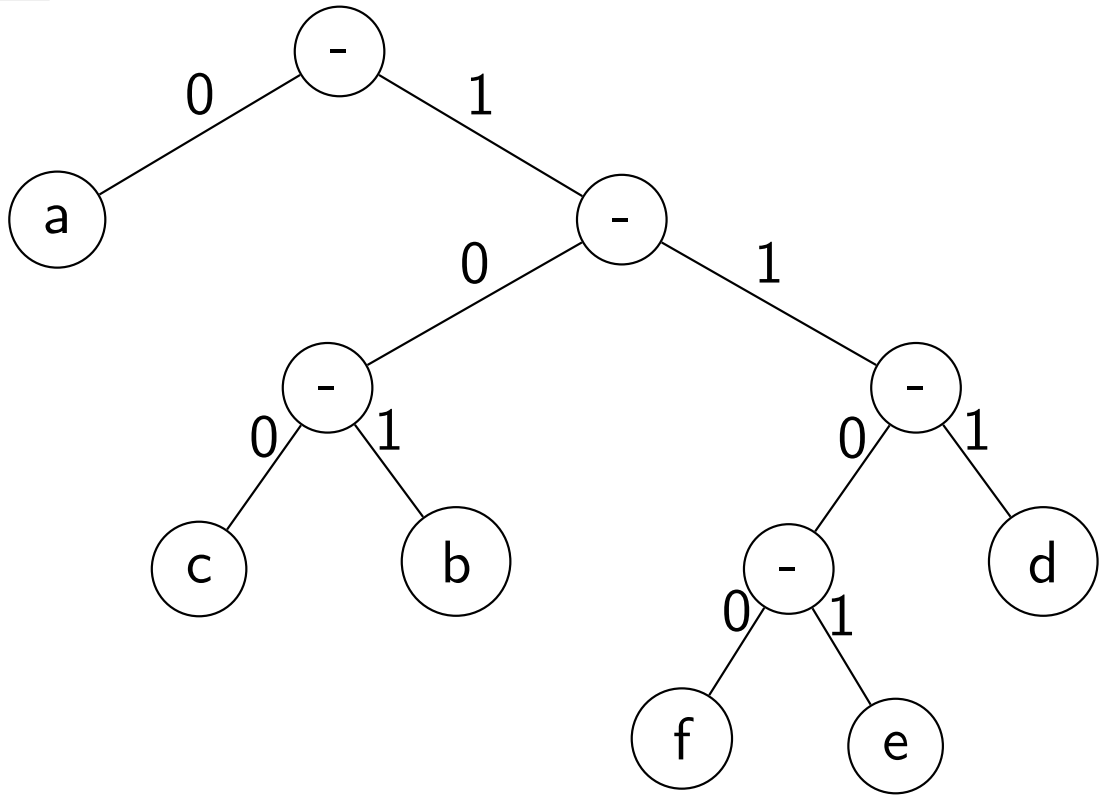
a,b,c,d,e,f	100



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Huffman(f[1..n], c[1..n])
  // PQ is a priority queue of (Node, freq) tuples
  for i ← 1 to n:
    t ← new Node(c[i],null,null)
    PQ.insert((t, f[i]))
  for i ← 1 to n-1
    (t1, f1) ← PQ.extractMin()
    (t2, f2) ← PQ.extractMin()
    t ← new Node('-',t1,t2))
    PQ.insert((t,f1+f2))
  return PQ.extractMin()
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Before we do that we first show that:

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- WLOG, assume that y is the left child of x, corresponding to binary 0.
- We can obtain another tree by deleting the link (x, y) and replacing node x by the subtree rooted at y.
- That turns out to be a better tree than the original, which is a contradiction.

The greedy choice property

Lemma

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This lemma tells us that it is a "safe" first choice to merge the symbols with the lowest frequencies into a sub-tree.

So our "greedy choice" of choosing the two smallest frequencies to merge was valid.

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The proof idea is to take the tree T representing an arbitrary optimal prefix-free code and modify it to make a tree representing another optimal prefix-free code such that the characters x and y appear as sibling leaves of maximum depth in the new tree.

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If we can do this, then their codes will have the same length and differ only in the last bit.

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- Let $f(x)$ be the frequency of character x.
- Without loss of generality, assume that $f(b) \leq f(c)$ and $f(x) \leq f(y)$. Since $f(x)$ and $f(y)$ are the two lowest frequencies, and $f(b)$ and $f(c)$ are arbitrary frequencies, we know that $f(x) \leq f(b)$ and $f(y) \leq f(c)$.

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- Without loss of generality, assume that $f(b) \leq f(c)$ and $f(x) \leq f(y)$. Since $f(x)$ and $f(y)$ are the two lowest frequencies, and $f(b)$ and $f(c)$ are arbitrary frequencies, we know that $f(x) \leq f(b)$ and $f(y) \leq f(c)$.
- We swap the positions of x and b in tree T to get tree T' and swap the positions of y and c in T' to get the tree T''.

The greedy choice property

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- We swap the positions of x and b in tree T to get tree T' and swap the positions of y and c in T' to get the tree T''.
- The difference in cost between the trees T and T' is:

$$B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c)$$

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B(T) - B(T') &= \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) \\
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A similar argument shows that $B(T') - B(T'') \geq 0$.

Thus $B(T) \geq B(T'')$. But since T is optimal, it must be the case that $B(T) = B(T'')$ and so T'' is optimal too.

Optimal substructure

Lemma

Let x and y be sibling leaves in an optimal tree T for C. Replace x and y by node z, and assign to z the frequency $f(z) = f(x) + f(y)$. Then $T' = T - \{x, y\}$ is an optimal tree for $C' = C - \{x, y\} \cup \{z\}$.

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In other words: the optimal solution T to the prefix-free code tree problem can be obtained by combining the greedy choice with the solution T' to the problem in which x and y have been replaced by a node z whose frequency is $f(z) = f(x) + f(y)$.

Optimal substructure

- We start by showing that the cost of the tree T can be expressed in terms of the cost of the tree T'.
- For each $c \in C - \{x, y\}$, we know that $d_T(c) = d_{T'}(c)$. This is because any leaf other than x or y is located at the same level in tree T' as in tree T.
- Thus $f(c)d_T(c) = f(c)d_{T'}(c)$ for every $c \in C - \{x, y\}$.
- Since $d_T(x) = d_T(y) = d_{T'}(z) + 1$, we know that

$$\begin{aligned} f(x)d_T(x) + f(y)d_T(y) &= (f(x) + f(y))(d_{T'}(z) + 1) \\ &= f(z)d_{T'}(z) + f(x) + f(y) \end{aligned}$$

- This means that $B(T) = B(T') + f(x) + f(y)$.

Optimal substructure

- Suppose that T' is not optimal for $C - \{x, y\} \cup \{z\}$.
- That means that there is a tree T'' whose leaves are characters in $C - \{x, y\} \cup \{z\}$ such that $B(T'') < B(T')$. Since z is a character in $C - \{x, y\} \cup \{z\}$ it must appear as a leaf in the tree T''.
- So by adding x and y as children of z, we obtain a prefix-free codes for C with cost $B(T'') + f(x) + f(y) < B(T') + f(x) + f(y) = B(T)$.
- This contradicts the optimality of T.
- So it must be the case that T' is optimal for $C - \{x, y\} \in \{z\}$.

Resident Match Problem

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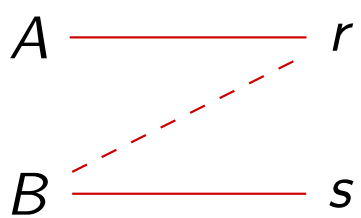

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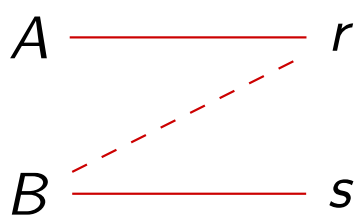

We call (r, B) an unstable pair for the matching.

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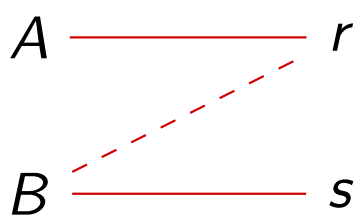

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The goal of the Resident Match Problem is a matching with no unstable pairs, i.e. a stable matching.

Resident Match Problem

3 doctors q, r, s and 3 hospitals A, B, C rank each other:

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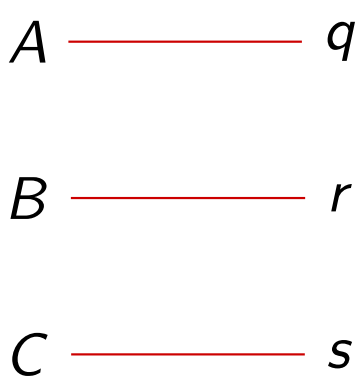
The matching $\{A - q, B - r, C - s\}$



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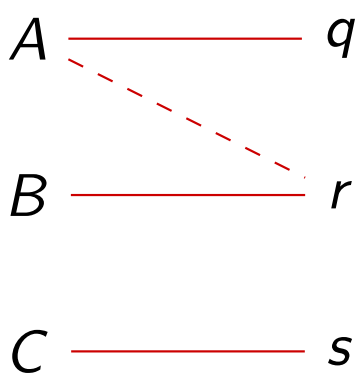
The matching $\{A-q, B-r, C-s\}$ is unstable, because A would rather hire r than q, and r would rather work at A than at B.



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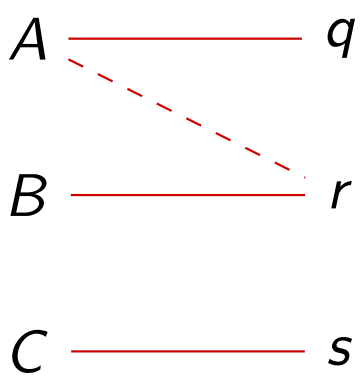

(A, r) is an unstable pair for this matching.

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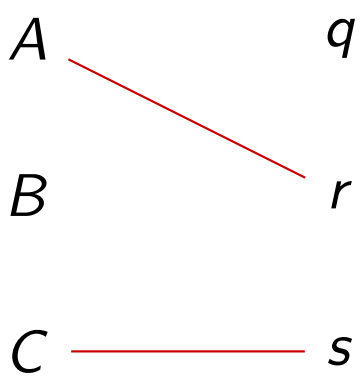

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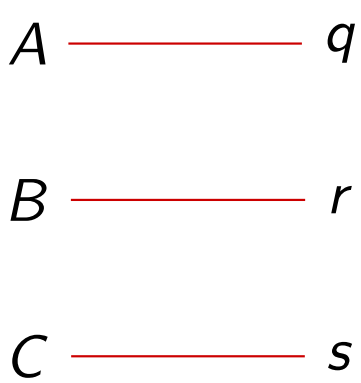

How about we greedily fix the unstable pairs?

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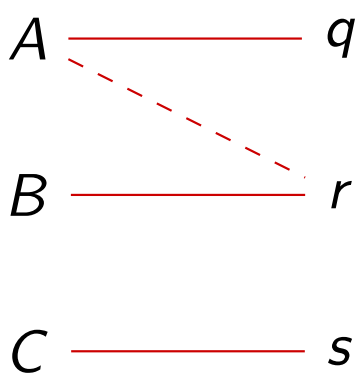

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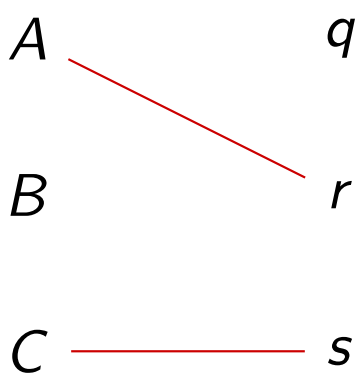

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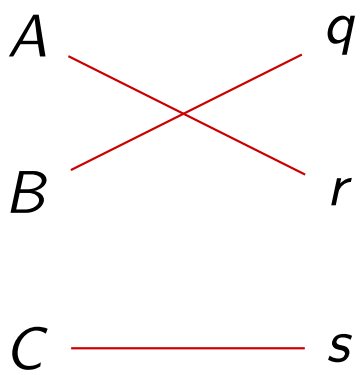

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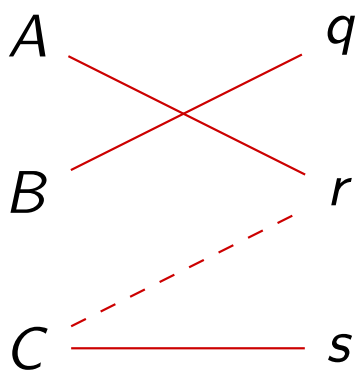

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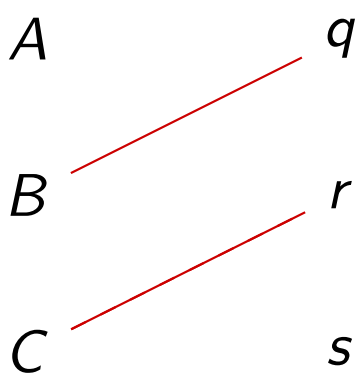

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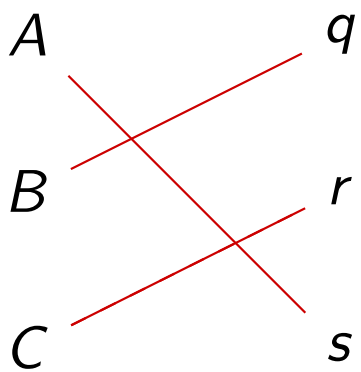

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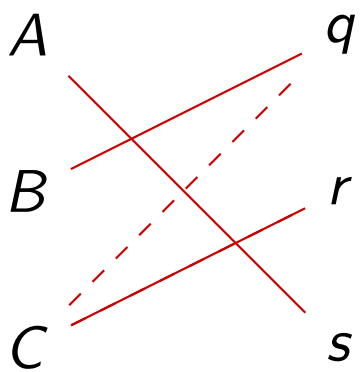

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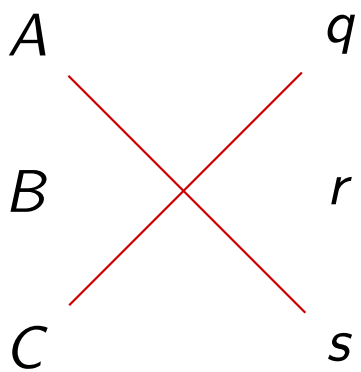

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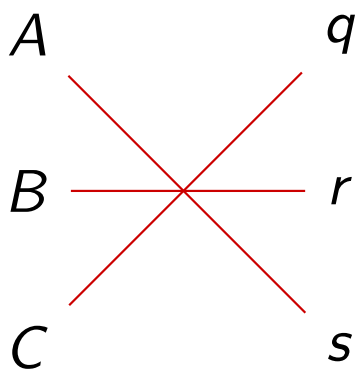

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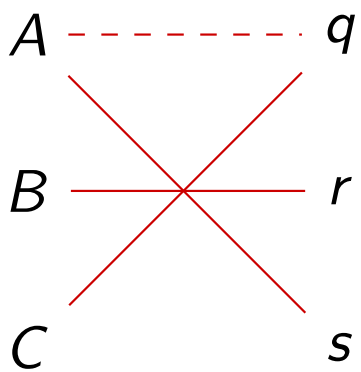

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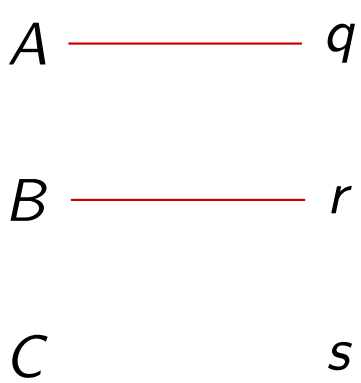

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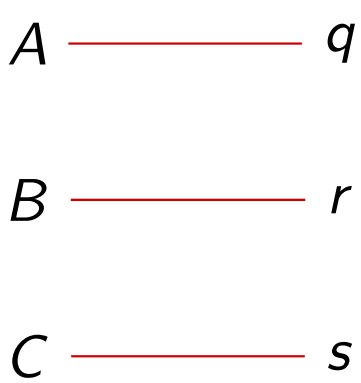

How about we greedily fix the unstable pairs?

A wrong greedy approach?

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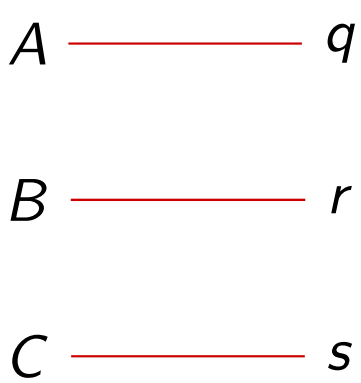

How about we greedily fix the unstable pairs? Nope... Back to where we started.

Gale-Shapley Stable Matching algorithm

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Hospitals make offers greedily, doctors accept offers greedily.

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	A	B	C	D
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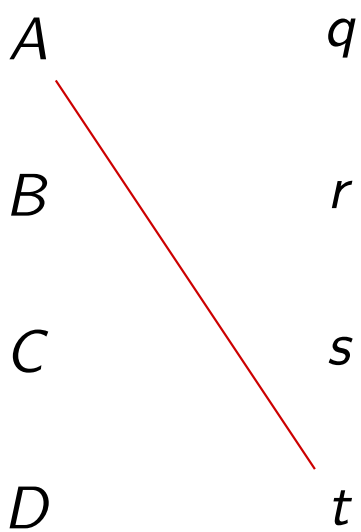

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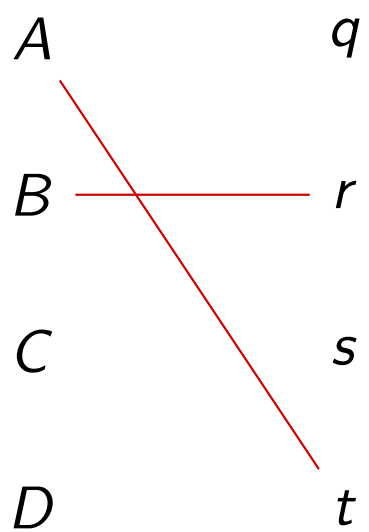

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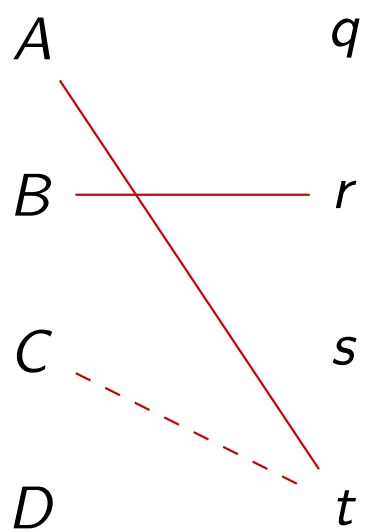

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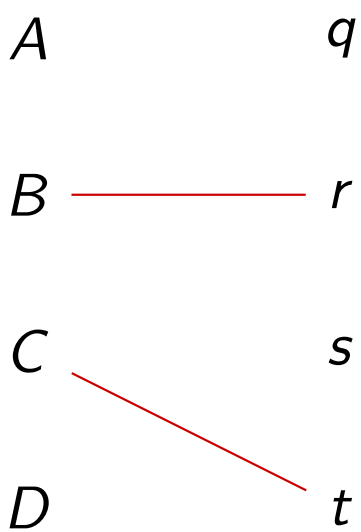

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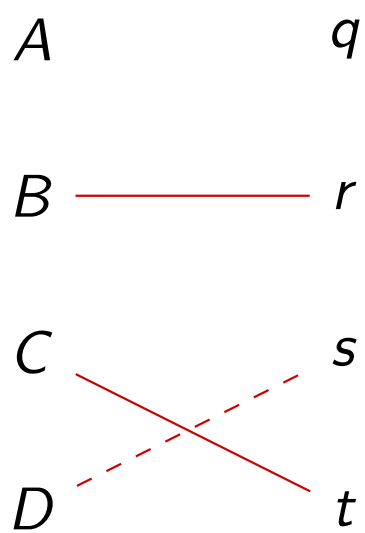

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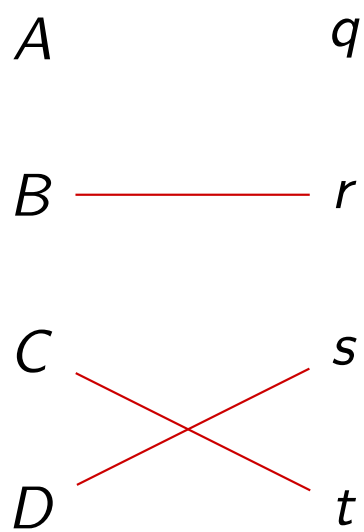

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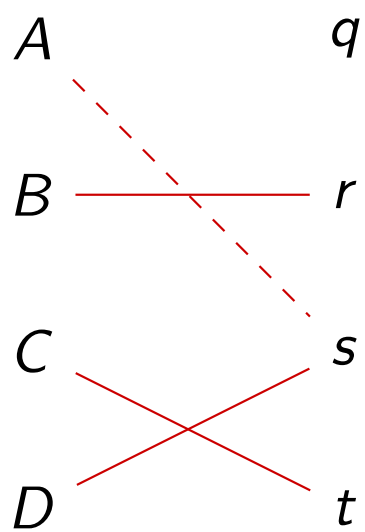

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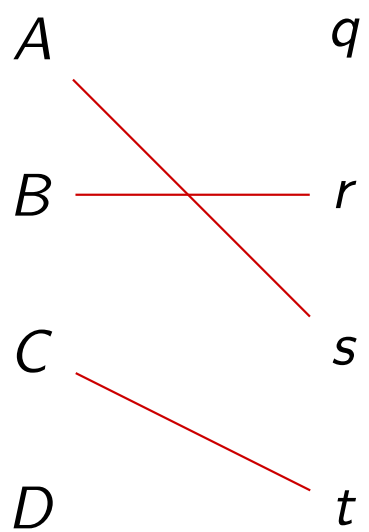

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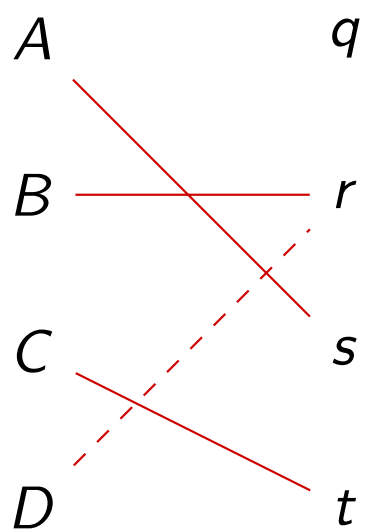

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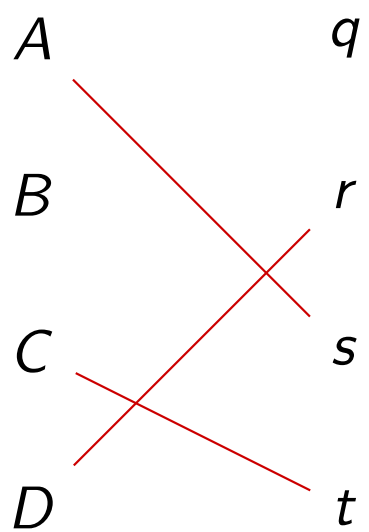

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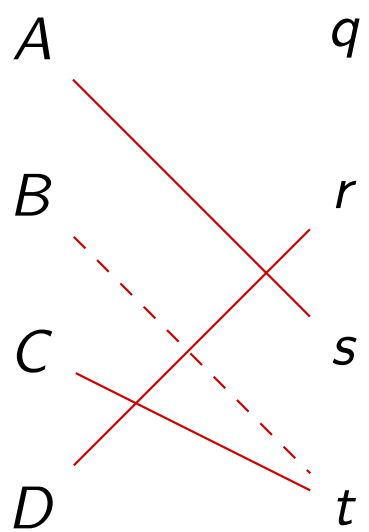

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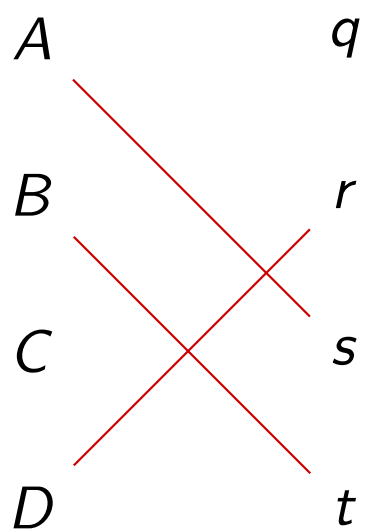

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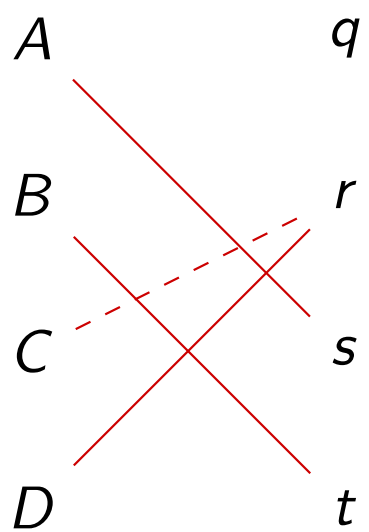

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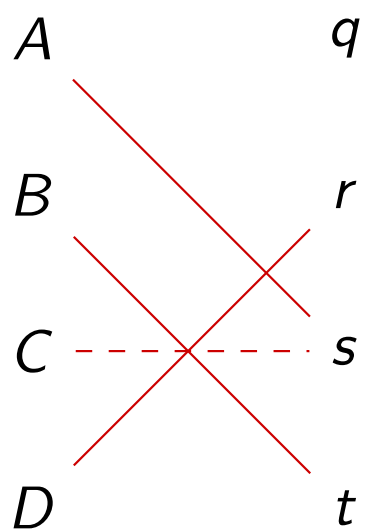

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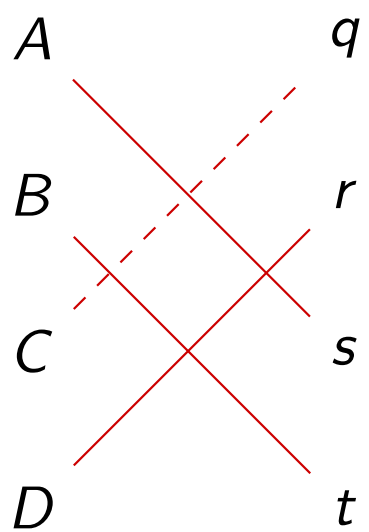

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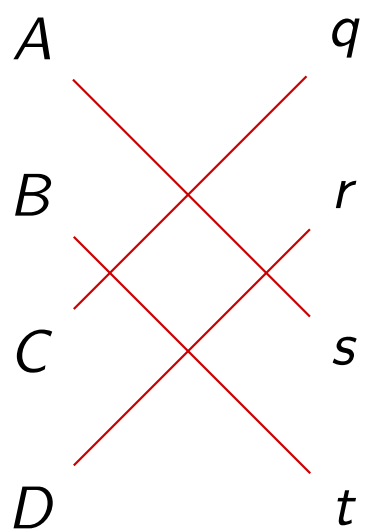

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In other words, the algorithm always computes a perfect matching between doctors and hospitals.

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We conclude that there are no unstable pairs; the matching is stable!