2-D Geometry for Programming Contests[1]

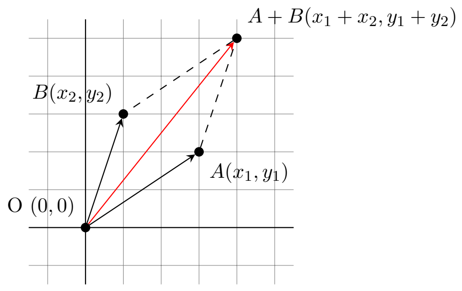
1 Vectors

A vector is defined by a direction and a magnitude. In the case of 2-D geometry, a vector can be represented as a point $A = (x, y)$, representing the vector from origin $O = (0, 0)$ to A, which gives both a direction and a magnitude. In practical examples, vectors with starting points other than origin O, such as the vector from $(1, 3)$ to $(5, 1)$, can be represented by the "translated" vector $(5-1, 1-3) = (4, -2)$. This means that $(4, -2)$ captures the direction and magnitude of the vector from $(1, 3)$ to $(5, 1)$ and thus can be used in its place. In general, the vector from point $A = (x_1, y_1)$ to point $B = (x_2, y_2)$ can be represented by the vector $A - B = (x_2 - x_1, y_2 - y_1)$.

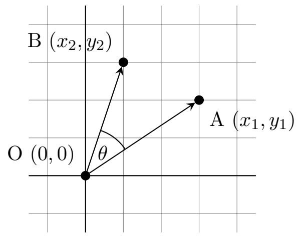
In order to keep the presentation that follows simple, we will deal with vectors starting at the origin $(0, 0)$ and represented using a single point $A = (x, y)$. The pseudo-code, however, will work with vectors not necessarily starting at the origin.

2 Vector Addition

There are a number of mathematical operations that can be performed on vectors. The simplest of these is addition: you can add two vectors together and the result is a new vector. If you have two vectors $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the sum of the two vectors is simply $A + B = (x_1 + x_2, y_1 + y_2)$.



Vector addition



Dot product

3 Dot product

The dot product of vectors $A = (x_1, y_1)$ and $B = (x_2, y_2)$, denoted by $A \cdot B$, is $x_1 * x_2 + y_1 * y_2$. Note that this is not a vector, but is simply a single number (called a scalar). The reason the dot product is useful is because $A \cdot B = |A||B|\cos(\theta)$, where θ is the angle (less than 180 degrees) between vectors A and B. $|A|$ is called the norm of the vector, and in a 2-D geometry problem is simply the length of the vector, $\sqrt{x_1 + y_1}$. Therefore, we can calculate $\cos(\theta) = (A \cdot B)/(|A||B|)$. By using the arccos (or, $\cos^{-1}$) function, we can then find θ. It is useful to recall that $\cos(90) = 0$ and $\cos(0) = 1$, as this tells you that a dot product of 0 indicates two perpendicular lines, and that the dot product is greatest when the lines are parallel. The following pseudo code computes the dot product of vectors AB, going from point A to point B, and AC, going from point A to point C:

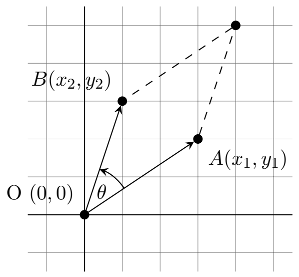
[1] Adapted from http://www.topcoder.com/community/data-science/data-science-tutorials/geometry-concepts-basic-concepts/

```
//Compute the dot product AB  AC
int dot(int[] A, int[] B, int[] C) {
    AB = new int[2]; AC = new int[2];
    AB[0] = B[0]-A[0]; AB[1] = B[1]-A[1];
    AC[0] = C[0]-A[0]; AC[1] = C[1]-A[1];
    int dot = AB[0] * AC[0] + AB[1] * AC[1];
    return dot;
}
```

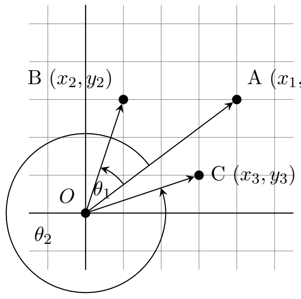
4 Cross Product

An even more useful operation is the cross product. The cross product of two vectors $A = (x_1, y_1)$ and $B = (x_2, y_2)$, denoted by $A \times B$, is $x_1 * y_2 - y_1 * x_2$. Note that $A \times B \neq B \times A$.

Similar to the dot product, $A \times B = |A||B|\sin(\theta)$. However, θ has a slightly different meaning in this case: θ is the *counterclockwise* angle *from OA to OB*. This means that OB is less than 180 degrees counterclockwise from OA iff $\sin(\theta)$ is positive; also, OB is more than 180 degrees counterclockwise from OA iff $\sin(\theta)$ is negative. The cross product can be used to determine whether two points C and D are on opposite sides of the line going through vector A: start by computing the cross products $A \times B$ and $A \times C$; if the cross products have different signs then then points B and C must lie on opposite sides of the line as shown in the figure below. Another useful



Cross product



Determining whether two points are on opposites sides of a line

fact related to the cross product is that the absolute value of $|A||B|\sin(\theta)$ is equal to the area of the parallelogram with two of its sides formed by A and B. Furthermore, the triangle formed by O, A, and B has half of the area of the parallelogram, so we can calculate its area from the cross product also.

```
//Compute the cross product AB x AC
int cross(int[] A, int[] B, int[] C) {
    AB = new int[2]; AC = new int[2];
    AB[0] = B[0]-A[0]; AB[1] = B[1]-A[1];
    AC[0] = C[0]-A[0]; AC[1] = C[1]-A[1];
    int cross = AB[0] * AC[1] - AB[1] * AC[0];
    return cross;
}
```

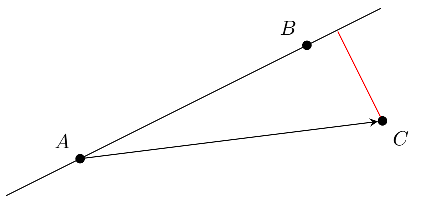
5 Segment-segment intersection

To determine whether two segments $[AB]$ and $[CD]$ interset, one needs to determine whether points C and D are on opposites sides of the line defined by segment $[AB]$ *and* that points A and B are on opposites sides of the line defined by segment $[CD]$. The first can be done by computing the cross products $AB \times AC$ and $AB \times AD$ and verifying that they have opposite signs. The second is done by computing the cross products $CD \times CA$ and $CD \times CB$ and verifying that they have opposite signs.

6 Line-Point Distance

Let's say that you are given 3 points, A, B, and C, and you want to find the distance from the point C to the line defined by A and B. The first step is to find the two vectors from A to B (AB) and from A to C (AC). Now, take the cross product $AB \times AC$, and divide by $|AB|$. This gives you the distance (illustrated below with the red line) as $(AB \times AC)/|AB|$.

The reason this works comes from some basic high school level geometry. The area of a triangle is found as $base * height/2$. Now, the area of the triangle formed by A, B and C is given by $(AB \times AC)/2$. The base of the triangle is formed by AB, and the height of the triangle is the distance from the line to C. Therefore, what we have done is to find twice the area of the triangle using the cross product, and then divided by the length of the base. As always with cross products, the value may be negative, in which case the distance is the absolute value.



Line-Point distance

Things get a little bit trickier if we want to find the distance from a line segment to a point. In this case, the nearest point might be one of the endpoints of the segment, rather than the closest point on the line. In the diagram above, for example, the closest point to C on the line defined by A and B is not on the segment AB, so the point closest to C is B. While there are a few different ways to check for this special case, one way is to apply the dot product. First, check to see if the nearest point on the line AB is beyond B (as in the example above) by taking $AB \cdot BC$. If this value is greater than 0, it means that the angle between AB and BC is between -90 and 90, exclusive, and therefore the nearest point on the segment AB will be B. Similarly, if $BA \cdot AC$ is greater than 0, the nearest point is A. If both dot products are negative, then the nearest point to C is somewhere along the segment.

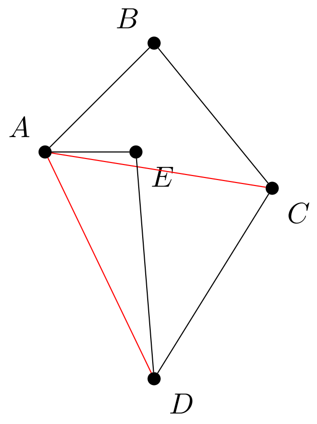
```
//Compute the distance from AB to C
//if isSegment is true, AB is a segment, not a line.
double linePointDist(int[] A, int[] B, int[] C, boolean isSegment){
    double dist = cross(A,B,C) / distance(A,B);
    if(isSegment){
        int dot1 = dot(A,B,C);
        if(dot1 > 0)return distance(B,C);
        int dot2 = dot(B,A,C);
        if(dot2 > 0)return distance(A,C);
    }
    return abs(dist);
}
```

Here is function `distance`:

```
//Compute the distance from A to B
double distance(int[] A, int[] B){
    int d1 = A[0] - B[0];
    int d2 = A[1] - B[1];
    return sqrt(d1*d1+d2*d2);
}
```

7 Polygon Area

Another common task is to find the area of a polygon, given the points around its perimeter. Consider the non-convex polygon on the right, with 5 points. To find its area we are going to start by triangulating it. That is, we are going to divide it up into a number of triangles. In this polygon, the triangles are ABC, ACD, and ADE. But wait, you protest, not all of those triangles are part of the polygon! We are going to take advantage of the signed area given by the cross product, which will make everything work out nicely. First, we'll take the cross product of $AB \times AC$ to find the area of ABC. This will give us a negative value, because of the way in which A, B and C are oriented. However, we're still going to add this to our sum, as a negative number. Similarly, we will take the cross product $AC \times AD$ to find the area of triangle ACD, and we will again get a negative number. Finally, we will take the cross product $AD \times AE$ and since these three points are oriented in the opposite direction, we will get a positive number. Adding these three numbers (two negatives and a positive) we will end up with a negative number, so will take the absolute value, and that will be area of the polygon.



Polygon area

The reason this works is that the positive and negative number cancel each other out by exactly the right amount. The area of ABC and ACD ended up contributing positively to the final area, while the area of ADE contributed negatively. Looking at the original polygon, it is obvious that the area of the polygon is the area of $ABCD$ (which is the same as $ABC + ABD$) minus the area of ADE. One final note, if the total area we end up with is negative, it means that the points in the polygon were given to us in clockwise order. Now, just to make this a little more concrete, lets write a little bit of code to find the area of a polygon, given the coordinates as a 2-D array, `p`.

```
int area = 0; int N = lengthof(p);
//We will triangulate the polygon into triangles with points p[0],p[i],p[i+1]
for(int i = 1; i+1<N; i++){
    int x_1 = p[i][0] - p[0][0];      int y_1 = p[i][1] - p[0][1];
    int x_2 = p[i+1][0] - p[0][0];    int y_2 = p[i+1][1] - p[0][1];
    int cross = x_1*y_2 - x_2*y_1;
    area += cross;
}
return abs(cross/2.0);
```

Notice that if the coordinates are all integers, then the final area of the polygon is one half of an integer.

8 Line-Line Intersection

One of the most common tasks you will find in geometry problems is line intersection. The first question is, what form are we given our lines in, and what form would we like them in? Ideally, each of our lines will be in the form $Ax + By = C$, where A, B and C are the numbers which define the line. However, we are rarely given lines in this format, but we can easily generate such an equation from two points. Say we are given two different points, (x_1, y_1) and (x_2, y_2), and want to find A, B and C for the equation above. We can do so by setting

$$\begin{aligned} A &= y_2 - y_1 \\ B &= x_1 - x_2 \\ C &= A * x_1 + B * y_1 \end{aligned}$$

Regardless of how the lines are specified, you should be able to generate two different points along the line, and then generate A, B and C. Now, let's say that you have lines, given by the equations:

$$\begin{aligned} A_1x + B_1y = C_1 \\ A_2x + B_2y = C_2 \end{aligned}$$

To find the point at which the two lines intersect, we simply need to solve the two equations for the two unknowns, x and y.

```
double det = A1*B2 - A2*B1       // determinant
if(det == 0){
    //Lines are parallel
}else{
    double x = (B2*C1 - B1*C2)/det
    double y = (A1*C2 - A2*C1)/det
}
```

To see where this comes from, consider multiplying the top equation by B_2, and the bottom equation by B_1. This gives you

$$\begin{aligned} A_1B_2x + B_1B_2y = B_2C_1 \\ A_2B_1x + B_1B_2y = B_1C_2 \end{aligned}$$

Now, subtract the bottom equation from the top equation to get

$$A_1B_2x - A_2B_1x = B_2C_1 - B_1C_2$$

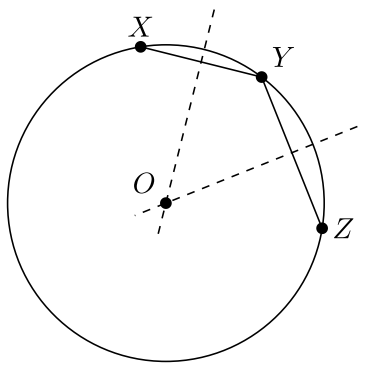
Finally, divide both sides by $A_1B_2 - A_2B_1$, and you get the equation for x. The equation for y can be derived similarly. Try it.

This gives you the location of the intersection of two lines, but what if you have line segments, not lines. In this case, you need to make sure that the point you found is on both of the line segments. If your line segment goes from (x_1, y_1) to (x_2, y_2), then to check if (x, y) is on that segment, you just need to check that $\min(x_1, x_2) \leq x \leq \max(x_1, x_2)$, and do the same thing for y. You must be careful about double precision issues though. If your point is right on the edge of the segment, or if the segment is horizontal or vertical, a simple comparison might be problematic. In these cases, you can either do your comparisons with some tolerance, or else use a fraction class.

9 Finding a Circle From 3 Points

Given 3 points which are not colinear (all on the same line) those three points uniquely define a circle. But, how do you find the center and radius of that circle? This task turns out to be a simple application of line intersection. We want to find the perpendicular bisectors of XY and YZ, and then find the intersection of those two bisectors. This gives us the center of the circle.

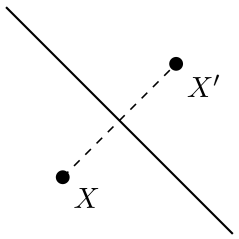
To find the perpendicular bisector of XY, find the line from X to Y, in the form $Ax+By=C$. A line perpendicular to this line will be given by the equation $-Bx+Ay=D$, for some D. To find D for the particular line we are interested in, find the midpoint between X and Y by taking the midpoint of the x and y components independently. Then, substitute those values into the equation to find D. If we do the same thing for Y and Z, we end up with two equations for two lines, and we can find their intersections as described above.



Finding a circle from 3 points

10 Reflection

Reflecting a point X across a line requires the same techniques as finding a circle from 3 points. First, notice that the distance from X to the line of reflection is the same as the distance from X', the reflection of X, to the line of reflection. Also note that the line between X and X' is perpendicular to the line of reflection. Now, if the line of reflection is given as $Ax+By=C$, then we already know how to find a line perpendicular to it: $-Bx+Ay=D$. To find D, we simply plug in the coordinates for X. Now, we can find the intersection of the two lines at Y, and then find $X'=Y-(X-Y)$.



Reflection

11 Rotation

Rotation doesn't really fit in with line intersection, but I felt that it would be good to group it with reflection. In fact, another way to find the reflected point is to rotate the original point 180 degrees about Y.

Imagine that we want to rotate one point around another, counterclockwise by θ degrees. For simplicity, lets assume that we are rotating about the origin. In this case, we can find that $x'=x\cos(\theta)-y\sin(\theta)$ and $y'=x\sin(\theta)+y\cos(\theta)$. If we are rotating about a point other than the origin, we can account for this by shifting our coordinate system so that the origin is at the point of rotation, doing the rotation with the above formulas, and then shifting the coordinate system back to where it started.

12 Convex Hull

A convex hull of a set of points is the smallest convex polygon that contains every one of the points. It is defined by a subset of all the points in the original set. One way to think about a convex hull is to imagine that each of the points is a peg sticking up out of a board. Take a rubber band and stretch it around all of the points. The polygon formed by the rubber band is a convex hull. There are many different algorithms that can be used to find the convex hull of a set of points. In this article, I'm just going to describe one of them, which is fast enough for most purposes, but is quite slow compared to some of the other algorithms.

First, loop through all of your points and find the leftmost point. If there is a tie, pick the highest point. You know for certain that this point will be on the convex hull, so we'll start with it. From here, we are going to move clockwise around the edge of the hull, picking the points on the hull, one at a time. Eventually, we will get back to the start point. In order to find the next point around the hull, we will make use of cross products. First, we will pick an unused point, and set the next point, N, to that point. Next, we will iterate through each unused points, X, and if $(X - P) \times (N - P)$ (where P is the previous point) is negative, we will set N to X. After we have iterated through each point, we will end up with the next point on the convex hull. See the diagram below for an illustration of how the algorithm works. We start with P as the leftmost point. Now, say that we have N and X as shown in the leftmost frame. In this case the cross product will be negative, so we will set $N = X$, and there will be no other unused points that make the cross product negative, and hence we will advance, setting $P = N$. Now, in the next frame, we will end up setting $N = X$ again, since the cross product here will be negative. However, we aren't done yet because there is still another point that will make the cross product negative, as shown in the final frame.

The basic idea here is that we are using the cross product to find the point which is furthest counterclockwise from our current position at P. While this may seem fairly straightforward, it becomes a little bit tricky when dealing with colinear points. If you have no colinear points on the hull, then the code is very straightforward.

```
convexHull(point[] X){
    int N = lengthof(X);
    int p = 0;
    //First find the leftmost point
    for(int i = 1; i<N; i++){
        if(X[i] < X[p])
            p = i;
    }
    int start = p;
    do{
        int n = -1;
        for(int i = 0; i<N; i++){

            //Don't go back to the same point you came from
            if(i == p)continue;

            //If there is no N yet, set it to i
            if(n == -1)n = i;
            int cross = (X[i] - X[p]) x (X[n] - X[p]);
            if(cross < 0){
                //As described above, set N=X
                n = i;
            }
        }
        p = n;
    }while(start!=p);
}
```

Once we start to deal with colinear points, things get trickier. Right away we have to change our method signature to take a boolean specifying whether to include all of the colinear points, or only the necessary ones.

```
//If onEdge is true, use as many points as possible for
//the convex hull, otherwise as few as possible.
convexHull(point[] X, boolean onEdge){
    int N = lengthof(X);
    int p = 0;
    boolean[] used = new boolean[N];
    //First find the leftmost point
    for(int i = 1; i<N; i++){
        if(X[i] < X[p])
            p = i;
    }
    int start = p;
    do{
        int n = -1;
        int dist = onEdge?INF:0;
        for(int i = 0; i<N; i++){
            //X[i] is the X in the discussion

            //Don't go back to the same point you came from
            if(i==p)continue;

            //Don't go to a visited point
            if(used[i])continue;

            //If there is no N yet, set it to X
            if(n == -1)n = i;
            int cross = (X[i] - X[p]) x (X[n] - X[p]);

            //d is the distance from P to X
            int d = (X[i] - X[p])  (X[i] - X[p]);
            if(cross < 0){
                //As described above, set N=X
                n = i;
                dist = d;
            }else if(cross == 0){
                //In this case, both N and X are in the
                //same direction.  If onEdge is true, pick the
                //closest one, otherwise pick the farthest one.
                if(onEdge && d < dist){
                    dist = d;
                    n = i;
                }else if(!onEdge && d > dist){
                    dist = d;
                    n = i;
                }
            }
        }
        p = n;
        used[p] = true;
    }while(start!=p);
}
```