New Bounds for the Firing Squad Problem on a Ring

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Keywords: Automata, algorithms, firing squad synchronization problem
Track: A

Abstract

We present two solutions to the firing synchronization problem on the ring, an 8-state minimal-time solution and a 6-state non-minimal-time solution. Both solutions use fewer states than the previous best-known minimal-time automaton, a 16-state solution due to Culik. We also give the first lower bounds on the number of states needed for solutions to the ring firing synchronization problem. We show that there is no 3-state solution and no 4-state, symmetric, minimal-time solution for the ring.

1 Introduction

In the original firing synchronization problem we consider a one-dimensional array of n identical finite automata. Initially all automata are in the same state except for one automaton that is designated as the initiator for the synchronization. The machines operate in lock-step, and the transitions of each automaton depend on the state of the automaton and the states of its neighbors. The goal is to define the set of states and transition rules for the automata so that all machines enter a special fire state for the first time and simultaneously during the final round of the computation. A great deal of work has been done on the original firing synchronization problem [1, 6, 10, 11, 12, 14]. It is interesting to note that the vast majority of this work has focused on finding solutions to the problem, and that few lower bounds are known for the original problem.

There are many variations of the firing synchronization problem that involve networks of automata other than the one-dimensional array [4, 5, 7, 8, 9, 12, 13]. We consider the problem of synchronizing rings of finite automata. In this problem each automaton has exactly two neighbors and there are no endpoints in the system. The goal is the same as the original problem, namely the synchronization of all automata in the final round of the computation.

Initial work on the ring variant of the firing synchronization problem focused on finding correct solutions to the problem without considering the number of states or even the minimal-time required to solve the problem [8, 5, 4, 9]. The main reason for this is that the solutions to the ring were given as an initial step in solving a more general problem, that of synchronizing connected graphs. The solution to the ring was not the goal, but a necessary first step. Further details about these solutions can be found in the next section.

The first work directly considering the number of states needed to solve ring synchronization was done by Culik [3]. He established the minimal time necessary for synchronization, n steps for a ring with n automata, and produced a minimal-time solution using 16 states. He did so in the context of solving a related synchronization problem in which there are multiple initiators.

Since the original interest in the firing squad problem on the ring was in finding solutions, minimal work has been done on finding state lower bounds for the ring. We are unaware of any results giving lower bounds for the firing synchronization problem on the ring prior to this work.

In this paper we improve Culik's result for the firing synchronization problem on a ring by giving an 8-state, symmetric, minimal-time solution. Not only does this solution use 8 fewer states than Culik's solution, it is symmetric. Few symmetric solutions to the firing synchronization problem exist [13]. We also give a 6-state, non-minimal-time solution for the ring, the first known non-minimal-time solution for the ring. Finally, we give the first state lower bounds known for the firing squad problem on the ring. We show that there is no 3-state solution, regardless of the time provided for synchronization. We also show that no 4-state, symmetric, minimal-time solution to the ring exists. These results are the first known lower bounds for the firing squad on the ring. Further, the 3-state bound is the first lower bound result for any version of the firing squad problem that makes no assumption about the time needed for synchronization. This provides a stronger bound, since it applies to both minimal-time and non-minimal-time solutions. The proofs of these results are also far simpler than those for any known lower-bound for any variant of the problem.

2 Preliminaries

We now outline the definitions for the ring version of the firing synchronization problem, sketch the previous work done on the problem, and state our results.

2.1 Definitions

One of the oldest variants of the firing synchronization problem is one in which the underlying network is not a one-dimensional array but a ring. As in the original firing problem there is a single initiator that may be located anywhere in the ring. The automata change state once during each round based on their current state and their neighbors' current state. The problem is to define the set of states and the transition function for the automaton so that all machines fire for the first time and simultaneously in some round $t(n)$.

The transition function for each automaton can be given as a set of 4-tuples. The 4-tuple (X,Y,Z,W) represents the rule that an automaton currently in state Y, with left neighbor in state X and right neighbor in state Z will enter state W at the next time step. We will denote this by XYZ $\rightarrow$ W. By definition automata solving the firing synchronization problem are deterministic so there is at most one 4-tuple (X,Y,Z,W) for any triple of states X,Y,Z.

A *symmetric* automaton is one which has a symmetric transition function, that is, whenever a transition XYZ $\rightarrow$ W is defined, the transition ZYX $\rightarrow$ W must also be defined. This means that the automata do not need to distinguish their left and right neighbors.

2.2 Previous work

Initial work on the synchronization of the ring was done while developing solutions for connected graphs. Each of the papers that follows provides a solution to the ring as a building block in a more general solution. The first result of this type was given by Nishitani and Honda [8] in 1981. Their solution required $3r_G + 1$ steps to synchronize a connected, undirected graph where r_G is the maximum distance between the initiator and any other node in the graph. Kobayashi [5] adapted Nishitani and Honda's solution to directed graphs, giving a solution requiring $(2a^2 + 1)^n$ time where a is the number of output terminals of the automaton and n is the number of nodes in the graph. The first polynomial-time solution for directed graphs was given by Even, Litman, and Winkler [4] in 1990. They found a solution requiring $O(n^2)$ time to synchronize a strongly-connected directed graph with n nodes. This was further improved by Ostrovsky and Wilkerson [9], who gave a solution that synchronized a strongly-connected directed graph in time $O(nD)$ where n is the number of nodes in the graph and D is the diameter of the graph. Note that none of the above work considers the number of states needed to synchronize the ring. The solutions are given at a high level, and transition functions are not provided.

The first work directly considering the number of states needed to solve ring synchronization was done by Culik [3]. In his paper Culik considered a variation of the firing synchronization problem in which there are multiple initiators. He showed that any solution to the problem for the one-dimensional array of length n with two initiators located at the endpoints requires $n - 1$ steps to synchronize. The following theorem is a direct corollary of this result.

Theorem 2.1 (Culik) *Any solution to the firing synchronization problem for the ring with n automata requires n time steps to synchronize.*

If we take an array of length $n + 1$ with two initiators at the endpoints and consider it as a ring with a single initiator we obtain Theorem 2.1. It should be noted that Culik incorrectly gave a time bound of $n - 1$ time steps in his paper, as he neglected to account for the fact that the length of the ring is shorter by one than the equivalent array since the two initiators are merged into one.

We will call any solution that synchronizes a ring of n automata in $t(n) = n$ time steps a *minimal-time solution*. Any solution that requires $t(n) > n$ time steps to synchronize will be called a *non-minimal-time solution*.

In addition to giving a time bound, Culik also described a minimal-time solution to the ring version of the problem. He used a modified version of Waksman's solution [14], producing an automaton that uses 16 states. To our knowledge, no prior work has been done on improving the number of states used by Culik's solution.

In understanding the history of work on lower bounds for the ring firing synchronization problem, it is instructive to consider lower bounds for the firing synchronization problem on the array. Despite the large body of work that exists for the firing synchronization problem, very little work has been done on finding lower bounds for the problem. The only known state lower bound for the problem was claimed in 1967 by

Balzer [1] and confirmed in 1994 by Sanders [10]. Sanders showed that there is no 4-state minimal-time solution to the restricted original problem, the problem where the initiator must be located at the right endpoint of the array.

The technique used for finding the state lower bound was a modified exhaustive search. Balzer wrote a simulation program that examined all possible 4-state solutions in an attempt to demonstrate that none of the solutions correctly solve the firing synchronization problem. Sanders showed that Balzer's program was incomplete but was able to write a correct implementation of the program to confirm the 4-state lower bound. It is crucial for both programs that only minimal-time solutions are examined. This is because the assumption allows the programs to discard any solution that uses more than $2n$-2 rounds for synchronize an array of n automata, making the search feasible.

Disappointingly, Sanders showed that this technique of exhaustive search does not scale. In particular, it cannot be used to determine if there is a 5-state solution to the firing synchronization problem on the array, as the search space grows too quickly. This leaves a gap between the best known solution, a 6-state automaton by Mazoyer [6], and the best known lower bound of 4-states. This problem has remained open for nearly 20 years.

In an attempt to tackle this long-standing open problem, we consider the number of states needed to solve the ring variant of the firing squad problem. It is our hope that by gaining insight into lower bounds for the ring, progress can be made in finding lower bounds on the number of states needed for the original problem. Finding a relationship between the structure of solutions for the ring and solutions for the array may aid us in producing the long-sought lower bound. To the best of our knowledge this avenue of research has not been explored, and no state lower bounds for the ring firing synchronization problem exist.

2.3 Our contributions

We present a 8-state, symmetric, minimal-time solution to the firing synchronization problem on the ring. This solution is adapted from Szwerinski's solution to the original firing synchronization problem [13]. To synchronize an array of length n the ring solution requires time n. It uses 8 fewer states than the best-known existing solution for the ring, and it is the first symmetric solution for the ring.

We also give a 6-state, non-minimal-time solution for ring synchronization. This solution is an extension of Mazoyer's solution to the original problem [6]. It requires $2n-2$ steps to synchronize an array with n automata. This is the first non-minimal-time solution for the firing synchronization problem on the ring, and it uses 2 fewer states that our minimal-time solution and 10 fewer states than Culik's solution.

We also give the first known state lower bounds for the firing synchronization problem on the ring. These are the first lower bound results for any variant of the firing synchronization problem that do not use exhaustive search. Our first state lower bound result is the following theorem:

Theorem 2.2 *There is no* 3*-state solution to the firing synchronization problem for the ring.*

As noted in the previous section, this is the first known lower bound for the firing synchronization problem that places no restrictions on the time required to synchronize. Such a result would not have been possible using previously existing techniques, as the exhaustive search programs depend on the fact that solutions requiring more than the minimal time can be discarded.

With two additional conditions, we can extend the theorem to the following:

Theorem 2.3 *There is no 4-state, symmetric, minimal-time solution to the firing synchronization problem for the ring.*

The ring provides an inherently symmetric setting, as the endpoints in the array are frequently the place where asymmetries are introduced in existing solutions to the firing synchronization problem. The simplest solutions to the ring synchronization problem often end up being symmetric. In fact, we conjecture that if an asymmetric, minimal-time solution exists for the ring, then there is a symmetric, minimal-time solution using the same number of states.

It should be noted that both of our theorems assume that a correct solution to the firing synchronization problem must synchronize all rings of length greater than or equal to 2. This assumption is motivated by the existing work on state lower bounds for the original firing synchronization problem. In fact, there are solutions to the firing squad problem on the array using only 4 states that work for small arrays. These semi-solutions synchronize arrays with 10 or fewer automata but fail to synchronize arrays of larger lengths [10]. Automata that work for only a subset of lengths are not considered genuine solutions.

3 The minimal-time solution

Our 8-state, minimal-time solution is adapted from Szwerinski's 8-state, symmetric solution to the firing synchronization problem on the one-dimensional array [13]. The construction of the solution requires the addition of some transitions to the solution, as well as the removal of transitions that are not needed for the ring, but the solution behaves in the same manner as Szwerinski's.

In Szwerinski's solution, the array is repeatedly subdivided into halves as new initiators are placed in the center(s) of each of the intervals. The simulation ends when all automata become initiators and then fire. The synchronization begins when the first initiator sends out a signal, the purpose of which is to produce a second initiator when it reaches the opposite end of the array. When this wake-up signal is reflected back by the new initiator it intersects with markers created in the wake of the first signal and produces a third initiator (or pair of initiators depending on the parity of the original array) located at the center of the array. This division of the array into halves continues until every other automaton is an initiator. At the next step in the simulation every automaton becomes an initiator, and at the next time step all automata fire. For a more detailed description of Szwerinski's solution, see his paper [13]. Important here is that the initiator state is G, Q and P serve as parity markers, B is a marker used to create new initiators, R is the state used to create and advance the B markers, Z is the quiescent state, and A serves as both the wake-up signal and the state that interacts with automata in state B to produce new initiators.

Because the eight-state solution is symmetric, it can be adapted to the ring in a straightforward manner. Instead of a single wake-up signal, two signals are sent from the initiator. These intersect on the opposite side of the ring, creating either one or two new initiators depending on the parity of the ring. The process then continues as described above until every automaton becomes an initiator and then fires.

To produce the above results on the ring, two types of changes to the transition function had to be made. First, all unnecessary transitions were eliminated. A transition is unnecessary if it involves a triple that does not appear in any simulation. Clearly, any transition involving the end marker is unnecessary, as the end marker is used in solutions to the original problem to indicate the end of the array. The marker allows the definition of a single transition function instead of three different types of transition functions, one for the central automata and one for each of the left and right end machines. Since there are no endpoints in the ring, these transitions can be removed. In addition, the transitions ARA $\rightarrow$ Q, PRQ $\rightarrow$ Q, QRP $\rightarrow$ Q, and QRQ $\rightarrow$ Q were eliminated. Each corresponds to a configuration produced only for arrays.

Next, additional transitions had to be defined for configurations that appeared in simulations on the ring but did not appear in any simulations in the array. These transitions are AZA $\rightarrow$ G, AAR $\rightarrow$ G, RAA $\rightarrow$ G, AAP $\rightarrow$ G, PAA $\rightarrow$ G, AAG $\rightarrow$ G, GAA $\rightarrow$ G, QGG $\rightarrow$ G, and GGQ $\rightarrow$ G. All of these configurations represent triples produced late in the simulation and are of two types. The first is triples produced immediately following the creation of the center initiator or initiators, and the second is triples that occur just prior to synchronization. The state A is used in Szwerinski's solution as a pseudo-initiator to break symmetry in these places in the simulation. Because the ring produces more symmetric behavior than the array, more transitions designed for this purpose were needed.

Table 1 shows the transition function for the 8-state automaton. The state of an automaton at the next time step can be found by looking at the entry in the column

neighbors' states	present state Z	A	B	R	P	Q	G
Z–Z	Z	Z	B				G
Z–A	A	Z	G	A			
Z–B	Z	G	B		P		
Z–R	R	P	P	Q	R	Z	G
Z–P	Z	R	B	Q			
Z–Q	Z		B	Q	R		
Z–G	A	R		B	A		G
A–A	G	G					G
A–B	A	G	G	G	P		
A–R	P	G				A	
A–P	R	G	G	Q			
A–Q	A		G		P		
A–G	R	G	G	B			G
B–B	Z				P		G

neighbors' states	present state Z	A	B	R	P	Q	G
B–R	R	P	P		R	Z	G
B–P	Z	R				Q	
B–Q	Z			B	R		
B–G	A	R	B		A		G
R–R		P					G
R–P		R	Q			Z	
R–Q		P				Z	G
R–G		R	B	A		A	G
P–P							A
P–Q	Z	R				Z	
P–G			B	A		A	A
Q–G	A	R			A		G
G–G	G		G	G			F

Table 1: The transition function for the 8-state automaton

corresponding to the automaton's present state and the row corresponding to the states of its neighbors. Since the automaton is symmetric, the orientation of the neighbors is irrelevant.

4 A non-minimal-time solution

The 6-state, non-minimal-time ring solution is an extension of Mazoyer's 6-state solution to the restricted firing synchronization problem [6]. The restricted version of the problem requires that the initiator to be located at the left endpoint of the array. The construction of a ring solution requires a slight modification of the transition function, but the solution behaves in the same manner as Mazoyer's. Since it was adapted from Mazoyer's solution, our solution requires $2n-2$ rounds to synchronize a ring of n automata. For a detailed description of Mazoyer's solution see his paper [6].

In order to extend Mazoyer's solution to the ring, two types of changes had to be made. First, all transitions involving the end marker were eliminated, as in with the 8-state solution described above. Then transitions were added to preserve the behavior of Mazoyer's solution. These transitions prevent the wake-up signal from propagating to the left of the first initiator and keep all of the initiators to the left of the first initiator quiescent. The transition function for the 6-state non-minimal-time automaton has been omitted for brevity. For a more detailed explanation of the changes to the automaton and the complete transition function, see the technical report [2].

The fact that a solution to the restricted firing synchronization problem could be adapted to work on a ring is remarkable. Particularly interesting in this case is that

Mazoyer's solution is distinctly non-symmetric. He relied on asymmetry to help him reduce the number of states needed for the solution, which is why the solution only works for the restricted version of the original problem. Despite this, the solution could be modified to work on the ring, where symmetry is inherent. We conjecture that this is a consequence of the structure of Mazoyer's solution, and that not all non-symmetric solutions can be modified for the ring.

5 Lower bounds

As mentioned previously, there are no known lower bounds for the firing synchronization problem on a ring. In this section we show that there is no 3-state solution to ring synchronization. We also show that there is no 4-state, symmetric, minimal-time solution.

5.1 Three-state bound

We now prove Theorem 2.2, a result stating that there is no 3-state solution to the firing synchronization problem on the ring.
Proof: Denote the three states by G, Z, and F. Without loss of generality we may assume that F is the firing state, G is the initiator state, and Z is the quiescent state. Since there are only three states for the solution and the fire state cannot be used prior to the final round, there are only eight possible triples of states that may be used prior to the last round. These are: ZZZ, GZZ, ZGZ, ZZG, GZG, GGZ, ZGG, and GGG. We know that ZZZ $\rightarrow$ Z must be defined by the definition of the firing synchronization problem. Partition the triples into four classes, based on the number of initiators.

Class 0	ZZZ
Class 1	GZZ, ZGZ, ZZG
Class 2	GGZ, GZG, ZGG
Class 3	GGG

Consider the ring of length 3. By assumption, the initial configuration is ZGZ. In order to produce the next configuration we must apply three class 1 rules. The next configuration, however, must have at least two initiators, since otherwise it would duplicate the initial configuration.

This means that there are two cases to consider:

1. Class 1 rules have all G's on the right hand side, or
2. Exactly two of the class 1 rules have a G on the right hand side.

In the first case, we must have GGG $\rightarrow$ F. This yields a contradiction for the ring of length four, where after one round we produce the configuration GGGZ.

In the second case, we must have all class 2 triples defined to have G on the right hand side, or we produce an infinite loop for the length three ring. This is because in the length three ring, the first configuration has two Z's and the second configuration has one Z. Since it is not possible by the definition of the problem to have three Z's, the next configuration must have no Z's.

So in the case where exactly two of the class 1 triples are defined to transition to G, all class 2 rules must use G on the right hand side. Further, we must have GGG $\rightarrow$ F must be defined, since it is the only remaining undefined triple. Consider the ring of length five. ZZZGZ yields ZZs, where s is a string with 2 initiators. If the initiators are adjacent in s, the we are done since in the next round we get the triple GGG from the triples ZGG, GGZ, and either ZZG or GZZ since two out of three of the class 1 triples are defined to transition to G. This produces a firing prior to the final round. If the string s is of the form GZG, so that the length 5 ring in round 2 looks like ZZGZG, we must have had ZGZ $\rightarrow$ Z, ZZG $\rightarrow$ G, and GZZ $\rightarrow$ G. This produces the configuration GGZGZ and then GGGZG, which causes a partial firing of the ring contrary to the definition of the problem. $\diamondsuit$

5.2 Four state bound

Recall that Theorem 2.3 states that there is no 4-state, symmetric, minimal-time symmetric solution to the ring synchronization problem. We now give the proof of the theorem.

Proof of Theorem 2.3

This result follows from the result due to Balzer [1] and Sanders [10] that there is no 4-state minimal-time solution to the firing synchronization problem on an array, and the following lemma:

Lemma 5.1 *If there exists a symmetric, minimal-time k-state solution to the firing squad problem on a ring with $2n-2$ automata, then there exists a symmetric, minimal-time k-state solution to the firing squad problem on an array of n automata.*

To see intuitively why this lemma is true, we describe the special case of how to construct a simulation of an array of $n = 6$ automata from a simulation on a ring of $2n-2 = 10$ automata. We first run a simulation on the ring, using the symmetric, minimal-time k-state solution to the firing synchronization problem on a ring:

0	G	Z	Z	Z	Z	Z	Z	Z	Z	Z
1	?	?	Z	Z	Z	Z	Z	Z	Z	?
2	?	?	?	Z	Z	Z	Z	Z	?	?
3	?	?	?	?	Z	Z	Z	?	?	?
4	?	?	?	?	?	Z	?	?	?	?
5	?	?	?	?	?	?	?	?	?	?
6	?	?	?	?	?	?	?	?	?	?
7	?	?	?	?	?	?	?	?	?	?
8	?	?	?	?	?	?	?	?	?	?
9	?	?	?	?	?	?	?	?	?	?
10	F	F	F	F	F	F	F	F	F	F

We obtain the simulation on an array of $n = 6$ automata by simply removing the last four columns.

In the arguments below, we will assume that $2n-2$ automata on a ring are numbered 1 through $2n-2$ in counter-clockwise order, with the initiator being numbered 1. Before we formally prove Lemma 5.1, we first show the following holds:

Claim 5.1 *Suppose we run a simulation of a symmetric solution on a ring of $2n-2$ automata. Then, in any round r, automata i and $2n-i$ must be in the same state, for $i = 2, 3, ..., n-1$.*

Proof: We use induction on r. If $r = 0$, the claim holds trivially since all relevant automata are quiescent. Consider now round $r \geq 1$ and choose some i between 2 and $n-1$. By induction, automata $i-1$ and $2n-i+1$ (or 1 if $i = 2$) are in the same state in round $r-1$, as are automata i and $2n-i$, and automata $i+1$ and $2n-i-1$. Since the solution is symmetric, this implies that automata i and $2n-i$ must be in the same state in round r. This is true for any i between 2 and $n-1$, which completes the induction step. $\diamondsuit$

We are now ready to complete the proof of Lemma 5.1.

Proof: From our example, it should be clear that all we need to do is define the additional transitions for the array solution that involve the left or right end markers. Let δ_l be the set of all the transitions of the k-state, symmetric, minimal-time ring solution that are used by automaton 1 in a ring of size $2n-2$, for any $n \geq 2$. Each transition in δ_l must be of the form $XYX \rightarrow W$, since automata 2 and $2n-2$ are always in the same state, by the above claim. For each such transition, we define a new, array transition $*YX \rightarrow W$. Next, we consider the set δ_r of transitions of the k-state, symmetric, minimal-time ring solution that are used by automaton n in a ring of size $2n-2$, for any $n \geq 2$. Each transition in δ_r is also of the form $XYX \rightarrow W$. For each such transition, we define a new array transition $XY* \rightarrow W$. $\diamondsuit$

It should be noted that the symmetric solution requirement in Theorem 2.3 and Lemma 5.1 is stronger than necessary. In the proof of Lemma 5.1, all we really used is that the following two conditions are satisfied by the k-state, minimal-time solution for the firing synchronization problem on a ring:

1. For any simulation on a ring of even length $2n-2$, automaton 1 does not use two transitions $X_1YZ \to W_1$ and $X_2YZ \to W_2$ where $X_1 \neq X_2$ and $W_1 \neq W_2$.
2. For any simulation on a ring of even length $2n-2$, automaton n does not use two transitions $XYZ_1 \to W_1$ and $XYZ_2 \to W_2$ where $Z_1 \neq Z_2$ and $W_1 \neq W_2$.

6 Conclusion

In this paper we presented improved bounds on the complexity of solutions to the firing synchronization problem on the ring. We gave an 8-state, symmetric, minimal-time solution, as well as a 6-state, non-minimal-time solution to the firing synchronization problem on the ring. Both of these solutions use fewer states that the best-known solutions to the firing synchronization problem on the ring, a 16-state solution given by Culik [3]. We also give the first lower bounds for the synchronization of the ring. We show that there is no 3-state solution to ring synchronization, the first known lower bound for any variant of the firing squad problem to place no restrictions on the time needed to synchronize. We also prove that there is no 4-state, symmetric, minimal-time solution. These results establish a lower bound for the firing synchronization problem on the ring. They are also the first lower bounds for any variant of the firing synchronization problem that do not rely on exhaustive search. The fact that the proofs are straightforward is a surprising bonus.

This work leaves a gap between the best-known upper bounds and lower bounds for ring synchronization. For minimal-time solutions this gap is 4 states in the symmetric case and 5 states in general. For non-minimal-time solutions the gap is only 3 states. Reducing this gap, either by producing a smaller solution for the ring or by improving the lower bounds, is an important direction for future work.

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